Optimal Sequence Method of Waiting Time Processes

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Abstract

Sequence / scheduling is the process of "turning arrivals and /or departure of units to their required service". It is used to refer to the "order" in which problems are solved. Some types of scheduling are considered, the formula for determining the total elapsed time derived and relevant theorems stated and proved. It is also shown that with any sequence S_0, the optimal sequence S^* can be obtained by the successive interchanges of consecutive jobs. For such interchange each value of D_n (s) is smaller than or equal to the one preceding the interchange.

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Introduction

A waiting-time problem concerns the determination of when and how to accomplish several jobs/tasks for the least cost and in minimal time. Waiting time is the length of time between the time a job is ready and the beginning of processing a job (Anderson, 1988). This type of problem arises when either units requiring services or facilities which are available for providing stands are idle. The problem could be one involving arrival which are randomly spaced and for service time of random duration. This class of problem includes situations requiring either the determination of optimal number of service facilities or the optimal arrival rate. The class of models seeking the solution of these facilities and scheduling problem is called waiting time theory. Scheduling problems involve solving for optimal schedule under various objectives, different machines, environment and characteristics of the job (Chikwendu 2011). The general sequencing problem aims to find the sequence out of $(n!)^m$ possible sequences which minimize the total elapsed time between the starting of the job in the first machine and the completion of the last job in the last machine (Frederich and Lieberman, 2008). The objective of scheduling according to (Samuel and Chowdhy, 1977) is to determine a production schedule which will meet all future demands at minimum total cost.

Waiting –time problem involves the determination of the amount of facilities that would minimize the "sum of cost associated with both customers and facility". The associated problem where facilities are fixed and arrival /sequence of having one subject to control entails the minimization of the pertinent cost. Minimizing waiting time variable (WTN) is a job scheduling problem where we schedule a batch of n-jobs for service on a single source in each machine. The machine can process at most one job at a time (Nongye *et al*., 2007). The problem involving the determination of the number of service facility required and timing of arrival is the queue theory. Sequencing theory is applicable to processes required to determine the order in which units available for receiving service should be served. The sequencing problem basically attempts to answer the question "in what sequence and where should the production lots be started" to minimize time wastage?

Taha (2006) writes that assignment problems involve scheduling workers to job on a one-to-one basis. The number of workers is presumed equal to the number of jobs, a condition that can be guaranteed by creating either fictitious workers or jobs as needed. He further stated that assignment problems can be converted into transportation problem by considering the workers as sources and the jobs as destinations, where all supplies and demands are equal. Martek (2021) considered the classical problem of scheduling n-tasks with given processing time on m-identical parallel processors so as to minimize completion time of a task. He used the lower bounds approximation algorithm and a branch and bounds procedure for the exact solution of the problem.

Types of sequencing model

There are many types of sequencing model but this paper will consider only two. The types considered here are;

(i) Two stations and n-jobs no passing.
(ii) Three stations and n-jobs no passing.

The use of Gantt chart will be helpful in this work.

(i) Two stations and n-jobs no passing: This type describes a situation where there are n-jobs and they are to be processed on two machines with each job requiring the same sequence of operation and no passing is allowed. i.e. any job processed first on one machine is also first processed on the other. The assumption here are as follows;

a) The material could be held between workstations;

b) All jobs must first go to the first machine A (say) and then the second B (say)

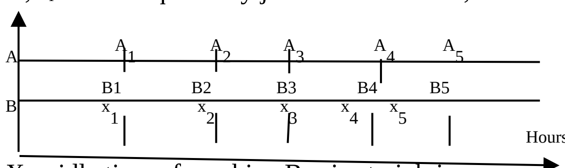
Now let A_i be the time required by job i on machine A,B_i be the required by job i on machine B,

And T the total elapsed time for the job 1, 2, …, n.; Also, let X_i be idle time of machine B from the end of job i-1 to start of job i.

Our expectation is a sequence (i_1, i_2, i_3…,i_n) where (i_1, i_2, i_3, ... , i_n) is the permutation of integers 1, 2, … n, which will minimize T.

Now take n=5 and represent it on a Gantt chart i.e.

Machine



X_i = idle time of machine B prior to job i

Determination of Total Elapsed Time (T)

Total elapsed time is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines. This is determined by the point of time at which job 1 goes on machine A and the point of time at which job 5 comes off machine B. Here at any instant of time, machine B is either working or idle. The total time machine B has to work is

$$\sum_{i=1}^{5} B_i$$

Now T = $\sum_{i=1}^{5} B_i$ + $\sum_{i=1}^{5} X_i$ ------------ (1)

But $\sum_{i=1}^{5} B_i$ is fixed,

:. The task of minimizing T reduces to that of minimizing $\sum_{i=1}^{5} X_i$ only

From the Gantt chart above, it is obvious that

$X_1 = A_1$

$X_2 = A_1 + A_2 - (B_1 + X_1)$, if $A_1 + A_2 \geq X_1 + B_1$

$X_2 = 0$ if $A_1 + A_2 < X_1 + B_1$

The expression for X_2 can be written as

$X_2 = \max(A_1 + A_2 - B_1, X_1, 0) = \max\left(\sum_{i=1}^{2} A_i - \sum_{i=1}^{1} B_i - \sum_{0}^{1} X_i , 0\right)$

Similarly $X_1 + X_2 = \max(A_1 + A_1 - B_1 - X_1)$

$= \max\left(\sum_{i=1}^{2} A_i - \sum_{i=1}^{1} (B_i, A_i \right)$

Also, $X_3 = A_1 + A_2 + A_3 - (B_1 + B_2 + X_1 + X_2)$

$= X_2 + A_3 - (B_2 + X_2) = A_3 - B_2$

$X_4 = A_1 + A_2 + A_3 + A_4 - (B_1 + B_2 + B_3 + X_1 + X_2 + X_3)$

$= X_3 + A_4 - (B_3 + X_3) = A_4 - B_3$

:. $X_3 = \max\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i - \sum_{i=1}^{2} X_{i,} 0\right)$

$\sum_{i=1}^{3} X_i = \max\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i - \sum_{i=1}^{3} X_{i,} 0\right)$

Now let $D_n(s) = \sum_{i=1}^{n} X_i$, where $D_n(s)$ is a function of the sequence.

Generally then,

$$D_n(s) = \sum_{i=1}^{n} X_i = \left(\sum_{i=1}^{n} A_{i-} \sum_{i=1}^{n-1} B_i - \sum_{i=1}^{n} X_i , 0\right)$$

$$= \sum_{i=1}^{n-1} X_i - \sum_{i=1}^{n-2} B_i , \dots \dots A_i$$

$$= \max_{1 \leq u \leq n} \left(\sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i\right) \text{ ---------------------- (2)}$$

ii) Three Station and n-jobs (no passing)

This procedure is similar to that in 1 above, except that three stations are involved. An extension of the notation for 2 above is as follows;

A_i = Time required by I on machine A

B_i = Time required by I on machine B

C_i = Time required by I on machine C

T = Total elapsed time for jobs 1, 2 ,3 … n.

X_i = Idle time on machine B from end of job I-1 to start of job i
Y_i = The idle time of the third machine before it starts work on the i^{th} job
Hence T = $(\sum_{i=1}^{n} C_i + \sum_{i=1}^{n} Y_i)$ --------------------------- (3)
$\sum_{i=1}^{n} C_i$ is fixed and hence
T_n => T => $\sum_{i=1}^{n} Y_i$
The optimum solution of this problem for the case where
i) Min $A_i \geq$ max B_i *or*
ii) Min $C_i \geq$ max B_i
is given by
$\sum_{i=1}^{n} Y_i = \max_{1\leq u\leq n}(H_v + K_u)$
Where $H_v = \sum_{i=1}^{u} B_i - \sum_{i=1}^{n} C_i, \quad u = 1,2,\dots n$
$K_u = \sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i$, u=1,2,…n

4. Theorem

The optimal sequencing decision rule for n jobs on two machines is given by Min (A_j, B_{j+1}) = Min (A_{j+1}, B_j)

Proof

Let there be two machines A and B with each machine requiring the same sequence of operations and no passing is allowed.

Let A_i = Time required by job i on machine A
B_i = Time required by job i on machine B
X_i = Idle time on machine B from the end of job i-1 to start of job i
$D_n(S) = \sum_{i=1}^{n} X_i$ = Total idle time on machine B for the sequence
Our need is to find sequence S^* of jobs (1, 2,...,n) such that
$D_n(S) \leq D_n(S_0)$ for any S_0
Now optimum sequence is got by the following rules.
J precedes job j+1 if Min $(A_j, B_{j+1}) \leq$ Min (A_{j+1}, B_j) and
Job j is indifferent to job j+1 (I.e. either precedes the other) If min (A_j, B_{j+1}) = Min (A_{j+1}, B_j)
This rule is proved as follows;
Start with a sequence $S^/$
Then obtain another sequence $S^{//}$ by interchanging the jth and the $(j+i)^{st}$ jobs i.e.
$S^/$=1, 2, 3,…, j-1, j, j+1, j+2,…,n.
$S^{//}$ = 1, 2,3,…, j-1, j, j+1, j+2,…,n.
Let $K_u = \sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i$ and
Let $K_u^/$ represent the K_u value of $S^/$ and
$K_U^{//}$ represent the K_u value of $S^{//}$
But $D_n(S) = \max_{1\leq u\leq n}(\sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i)$
$D_n(S) = \max_{1\leq u\leq n} K_u$
Then
$K_u^/ = K^{//}_u$ for u = 1,2,…, j-1, j-2, … , n
But $K^/_j$ and $K^/_{j+1}$ need not be equal to $K^/_{j+1}$ and $K_j^/$
Now consider two statements.
a) If max $(K^/_j, K^/_{j+1})$ = max $(K^{//}_j, K^{//}_{j+1})$ then,
$D_n(S^/) = D_n(S^{//})$
∴ Relative to the criterion of minimization, it makes no difference which sequence is used.
b) If however max $(K^/_j, K^/_{j+1}) <$ max $(K^{//}_j, K^{//}_{j+1})$, then $S^/$ is preferable to $S^{//}$ i.e. job j should precede job j+1. The relationship involved in the statement (b) here can be rewritten by first expanding $K^/_j$, i.e

$K^/_j = \sum_{i=1}^{1} A_i - \sum_{i=1}^{j-1} B_i$
$K^{//}_{j+1} = \sum_{i=1}^{j+1} A_i - \sum_{i=1}^{j+1} B_i$

∴ max $(K^/_j, K^/_{j+1})$ = max $(\sum_{i=1}^{j} A_i - \sum_{i=1}^{j-1} B_i , \sum_{i=1}^{j+1} A_i - \sum_{i=1}^{j-1} B_i)$ ------------------------------ (4)

Similarly,

$K^{//}_j = \sum_{i=1}^{j-1} A_i + A_{j+1} - \sum_{i=1}^{j-1} B_i$
$K^{//}_{j+1} = \sum A_i - \sum B_i - B_{j+1}$

∴ max $(K^{//}_J, K^{//}_{j+1})$ = max $(\sum_{i=1}^{j+1} A_i - A_{j+1} - \sum_{i=1}^{j-1} B_i , \sum_{i+1}^{j+1} A_i - \sum_{i=1}^{j-1} B_i - B_{j+1})$ − − − − − (5)

Now, from (4) and (5) we have

$$\max(-A_{j+1}, -B_j) < \max(-A_j, -B_{j+1}) \quad (6)$$

Then,

$$\max(K'_J, K'_{j+1}) < \max(K''_j, K''_{j+1}) \quad (7)$$

Multiply (6) by -1 to get

Min $(A_{j+1}, B_{j+1}) >$ min (A_j, B_{j+1})

Min $(A_j, B_{j+1}) <$ min (A_{j+1}, B_{j+1})

Statement (b) can now be written as the following; job j precedes j+1 when min $(A_i, B_i) + 1 <$ min A_{j+1}, B_{j+1}

5. Illustration

The following data shows Molcos machine time in hours for five jobs and two machines. Obtain the optimum sequence for Molcos.

i	A_i	B_i
1	3	6
2	7	2
3	4	7
4	5	3
5	7	4

Solution

Examine the A_i's and B_i's and find the smallest value i.e. min (A_i, B_i)

i.e. $B_2 = 2$

Is the value in A or B column?

Schedule the job first in the machine, i.e. B

Hence job 2 goes last on machine A

Cross off the job so assigned and continue to repeat the procedure

If there are ties, break arbitrarily by choosing any one

To continue this illustration, after job 2, the next minimum value is 3 in A_1 and B_4 respectively

If job 1 goes on machine A first, then job 4 i.e. B_4 goes on next to the last

.'. 4 is the next minimum in A_3 and B_5

First take job 3 to A, and job 5 on third

Subsequently, A_4 follows and B_1 too

The optimum sequence is therefore

1, 3, 5, 4, 2

The Gantt chart for this sequence could be drawn as follows:

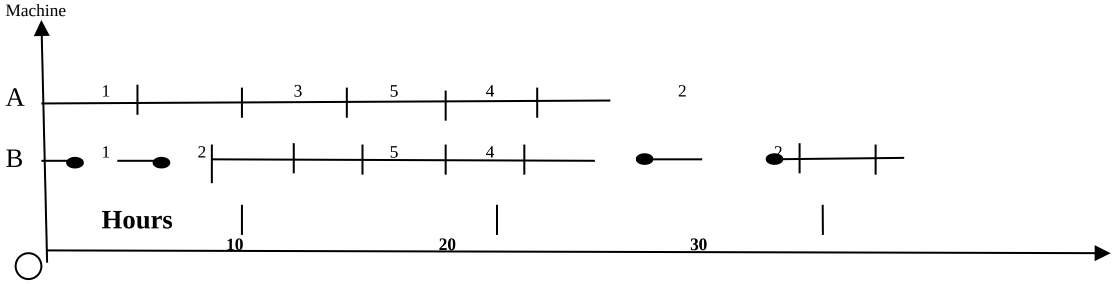


Illustration (ii)

The machine time (in days) for five jobs and machine A, B, C of Alaska Plc are as follows

i	A_i	$B_i C_i$
.1	8	5 4
2	10	6 9
3	6	28
.4	7	36
5	11	45

Obtain the optimal sequence for the company

Solution

Reduce the table summing A and B columns and B and C columns too, i.e.

i	$A_i + B_i$	$B_i + C_i$
1	13	9
2	16	15
3	8	10
4	10	9
5	15	9

The least value is 8 in row 3. The next is 9 in row 1,4 and 5 showing that 1,4 and 5 will come last in that order. The required optimal sequence is then

3, 2,1,4,5
3, 2, 4, 5,1
3, 2, 4, 1, 5

Summary

The theorem stated and proved shows that the result is transitive, the significance of this lies in the fact that starting with any sequence S_0, the optimal sequence S^* can be obtained by the successive interchange of consecutive jobs. That is, that there is always a pattern to be followed, each such interchange will give a value of $D_n(S)$ smaller than or the same as before the interchange.

Conclusion

Sequencing problem is one of the highly challenging problems in operations research. This paper has only exposed the general rules for some elementary cases. To minimize T, the total elapsed time for processing jobs, is a simple procedure as evolved by Johnson (1954) in introduction to operations research by Churchmen, C.W. The results show that there is a defined pattern that if used, leads to an optimal sequence. Obtaining an optimal sequence and applying it saves time and consequently enhances productivity.

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