Superlative Analysis of Mohand, Aboodh and Elzaki Transforms to the solution of a certain Partial Differential Equation with Financial Application

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Abstract

Mathematical problems which are advanced that can be seen in different fields of engineering and sciences can be solved by integral transforms. In this paper, the solutions to differential equations (DE) with emphasis on the stochastic differential equation (SDE) are achieved by implementing new integral transform namely; Aboodh, Elzaki and Mohand transforms and the transformation method and properties on the given DE. Furthermore, a superlative analysis of these transforms shows that the three transforms are well connected but the Elzaki transform showed a little distinction from others.to the solution of certain Partial differential equation with financial application.

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Introduction

Advanced problems of mathematics, physics and many other science disciplines can be solved using integral transforms like Aboodh transform, laplace transform, Mohand transform, Hankel transform and Elzaki transform to mention a few. Many real word problems which are mathematically represented by differential equations were solved by some scholars using these transforms.

Khalid Aboodh introduced the Aboodh transform (Aboodh, 2014) to facilitate the process of solving ordinary differential equations and partial differential equations in the time domain. Aboodh transform has been applied to solve ordinary differential equation with variable co-efficient by Mohand et al (2016). Aboodh et al (2016) obtained solution of partial integro-differential equations by applying and double Aboodh transformation method. Aboodh et al (2018) used Aboodh transformation method for solving delay differential equations.

Elzaki introduced the Elzaki transform (Elzaki, 2011) which is a modified Sumudu transform, a powerful tool applied to the solution of partial differential equations, ordinary differential equations etc. Elzaki *et al*. (2011) used the Elzaki transform to solve ordinary differential equations with variable coefficients. Aggrarwal *et al.* (2018) used Elzaki transform for solving population growth and decay problems. Aggrawal *et al.* (2018) applied Elzaki transform for solving linear volterra integral equations of first kind.

Mohand transform was introduced by Mohand (Mohand, 2018). Mohand transform is derived from the classical fouries integral; based on the mathematical simplicity of Mohand transform and its fundamental properties it was introduced to facilitate the process of solving ordinary and partial differential equations in the time domain. Aggarwal *et al*. (2018) solved the problems of population growth and decay problems by applying Mohand transform. Aggarwal *et al*. (2018) gave Mohand of Bessel's function; Kumar *et al.* (2018) gave the solution of linear volterra integro-differential equations using Mohand transform. This study looks at the superlative performance of the Aboodh, Elzaki and Mohand transform on a certain partial differential with financial implications.

Definition of Aboodh, Elzaki and Mohand Transforms

Definition of Aboodh Transform

Aboodh define "Aboodh transform" of the function $f(t)$ for $t \geq 0$ as

$A\{f(t)\} = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt = k(v), 0 < k_1 \leq v \leq k_2$, Where A is called the Aboodh transform operator.

Definition of Elzaki Transform

Elzaki, defined a new integral transform "Elzaki transform" of the function $f(t)$ for $t \geq 0\ as\ E\{f(t)\} = v\int_0^\infty F(t)e^{\frac{t}{v}}dt = T(v), 0 < k_1 \leq v \leq k_2$ where E is called the Elzaki transform operator.

Definition of Mohand Transform

Mohand and Mahgoub defined "Mohand transform" of the function $F(t) for\ t \geq 0\ as\ M\{F(t)\} = v^2\int_0^\infty F(t)e^{-vt}\,dt = R(v), 0 < k_1 \leq v \leq k_2$ where the operator M is called the Mohand transform operator.

Mohand, Aboodh, Elzaki transforms of the derivatives of the function F(t)

Mohand transform of the derivatives of the function $F(t)\ if\ M\{F(t)\} = R(v)$ then

$$M\{F'(t)\} = vR(v) - v^2F(0)$$
$$M\{F''(t)\} = v^2R(v) - v^3F(0) - v^2F(0)$$
$$M\{F^n(t)\} = V^nR(v) - v^{n+1}F(0) - v^nF(0) \ldots v^2F^{(n-1)}(0)$$

Aboodh transforms of the derivatives of the function $F(t) if\ A\{F(t)\} = K(v)$

$$A\{F'(t)\} = vk(v) - \frac{F(0)}{v}$$
$$A\{F''(t)\} = v^2K(v) = v^2K(v) - \frac{F'(0)}{v} - F(0)$$

S/N	$F(t)$	$K(v)A^{-1}\{F(t)\}$	$T(v)E^{-1}\{F(t)\}$	$R(v)M^{-1}\{F(t)\}$
1	1	$\frac{1}{v^2}$	v^2	v
2	t	$\frac{1}{v^3}$	v^3	1
3	t^2	$\frac{2!}{v^4}$	$2!\,v^4$	$\frac{2!}{v}$
4	$t^n, n > 1$	$\frac{t^n}{\lceil(n+1)}$	$\lceil(n+1)v^{n+2}$	$\frac{\Gamma(n+1)}{v^{n-1}}$
5	$t^n, n \in N$	$\frac{n!}{v^{n+2}}$	$n!\,v^{n+2}$	$\frac{n!}{v^{n-1}}$
6	e^{at}	$\frac{1}{v^2 - av}$	$\frac{v^2}{1 - av}$	$\frac{v^2}{v - a}$
7	$\sin at$	$\frac{a}{v(v^2 - a^2)}$	$\frac{av^3}{1 + a^2v^2}$	$\frac{av^2}{v^2 + a^2}$
8	$\cos at$	$\frac{1}{v^2 + a^2}$	$\frac{v^2}{1 + a^2v^2}$	$\frac{v^3}{v^2 + a^2}$

Application of Aboodh, Elzaki and Mohand transform to second order linear Black-Schole's differential equation.

Partial deferential equations (PDE.s) are used to model and analyse dynamic systems in fields as diverse as physics, biology, economics, and finance. The linear parabolic ones (LPDE.s) are one class of PDE.s which has received particular attention. The LPDEs make up a large class of PDE.s which is of a succinctly simple structure such that a thorough analysis of them is possible. In finance, for a contingent claim on a single asset, the generic PDE can be written as

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial^2 u}{\partial x^2}+b(x,t)\frac{\partial u}{\partial x}+c(x,t)u=0, \quad \ldots\ldots (1)$$

where t either represents calendar time or time-to-expiry, x represents either the value of the underlying asset or some monotonic function of it (e.g. $log(s_t)$; log-spot) and u is the value of the claim (as a function of x and t). The terms $a(.)$, $b(.)$ and $c(.)$ are the *diffusion*, *convection* and *reaction* coefficients respectively, and this type of PDE is known as a *convection-diffusion PDE*.2 This type of PDE can also be written in the form

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u}{\partial x}\right)+b(x,t)\frac{\partial}{\partial x}(\beta(x,t)u)+c(x,t)u=0. \quad \ldots\ldots (2)$$

This form occurs in the Fokker-Planck (Kolmogorov forward) equation that describes the evolution of the transition density of a stochastic quantity (e.g. a stock value). Equation (2) can be put in the form of equation (1) if the functions α and β are both once differentiable in x - although it is usually better to directly discretise the form given. A simple application in finance for this PDE can be found in [5].

A special case of interest in this work is the financial PDE that satisfies the following parabolic partial differential equation

$$\begin{cases}\frac{\partial \bar{u}}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 \bar{u}}{\partial s^2}+\alpha S\frac{\partial v}{\partial s}-rv=0, t>0,\\ v(S,0)=g(S).\end{cases} \quad \ldots\ldots (3)$$

In what follows, we obtain superlatively the solution of (3) by method of the aforementioned integral transforms.

Solution

Let the second order linear Black-Schole's differential equation (3) be given in the form (Osu and Sampson, 2018);

$$\frac{1}{2}\sigma^2 s^2\frac{d^2 v}{ds^2}+as\frac{dv}{ds}-rv=-s \ldots\ldots \quad (4)$$

Taking initial conditions; $v(0)=0, v'(0)=1$, then we have and by theorem 4.1 of (Osu and Sampson, 2018), equation (4) becomes;

$$\frac{d^2 v}{ds^2}+(\lambda_1+\lambda_2)\frac{dv}{ds}-\lambda_1\lambda_2 v=-s \ldots\ldots (5)$$

Aboodh transform

For this transform,

$$p^2 A(v)-\frac{v'(0)}{p}-v(0)+(\lambda_1+\lambda_2)\left[pA(v)-\frac{v(0)}{p}\right]-\lambda_1\lambda_2 A(v)=A(-s).$$

Which implies that

$$p^2 A(v)-\frac{v'(0)}{p}-v(0)+(\lambda_1+\lambda_2)\left[pA(v)-\frac{v(0)}{p}\right]-\lambda_1\lambda_2 A(v)=\frac{-1}{p^3},$$

so that

$$A(v)[p^2+p(\lambda_1+\lambda_2)-\lambda_1\lambda_2]=-\frac{1}{p^3}+\frac{v'(0)}{p}+v(0)+(\lambda_1+\lambda_2)\frac{v(0)}{p},$$

and

$$A(v)[p^2+p(\lambda_1+\lambda_2)-\lambda_1\lambda_2]=\frac{-1+p^2}{p^3}.$$

This implies that

$$A(V)=\frac{p^2-1}{p^3[p(\lambda_1+\lambda_2)-\lambda_1\lambda_2]}=\frac{p^2-1}{p^3(p-\gamma)(p-\beta)}=\frac{1}{p}\left[\frac{p^2-1}{p^2(p-\gamma)(p-\beta)}\right].$$

Resolving to partial fraction one gets.

$$\frac{p^2-1}{p^2(p-\gamma)(p-\beta)}=\frac{A}{p^2}+\frac{B}{(p-\gamma)}+\frac{C}{(p-\beta)}$$

So that for $p=0$, $A=-\frac{1}{\gamma\beta}$, for $p=\gamma$, $B=\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}$ and for $p=\beta$, $C=\frac{\beta^2-1}{\beta^2(\beta-\gamma)}$.

Therefore

$$\begin{aligned}\frac{1}{p}\left[\frac{p^2-1}{p^2(p-\gamma)(p-\beta)}\right]&=\frac{1}{p}\left[-\frac{1}{\gamma\beta p^2}+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)(p-\gamma)}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)(p-\gamma)}\right]\\&=-\frac{1}{\gamma\beta}\frac{1}{\rho^3}+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)\rho(\rho-\gamma)}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)\rho(\rho-\beta)}\\&=-\frac{s}{\gamma\beta}+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}e^{\gamma s}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)}e^{\beta s}.\end{aligned}$$

Where

$$\gamma = \frac{(-(\lambda_1+\lambda_2)+\sqrt{(\lambda_1+\lambda_2)^2}-4\lambda_1\lambda_2}{2}$$

$$\beta = \frac{(-(\lambda_1+\lambda_2)-\sqrt{(\lambda_1+\lambda_2)^2}-4\lambda_1\lambda_2}{2}$$

$$\beta - \gamma = \sqrt{\frac{1}{s^2} - \frac{4(s+\gamma)}{\sigma^2 s^2}}$$

$$\gamma - \beta = \sqrt{\frac{1}{s^2} - \frac{4(s+\gamma)}{\sigma^2 s^2}} \ , \ \ \gamma\beta = -s \ \ i.e. \ \gamma\beta = \lambda_1\lambda_2$$

Hence

$$v(s) = 1 + \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}e^{\gamma s} + \frac{\beta^2-1}{\beta^2(\beta-\gamma)}e^{\beta s}.$$

Elzaki transform

For this transform,

$$\frac{1}{\rho^2}T(\rho) - v(0) - \rho v'(0) + (\lambda_1+\lambda_2)\left[\frac{T(\rho)}{\rho} - \rho v(0)\right] - \lambda_1\lambda_2\, T(p) = T(-s)$$

from the Elzaki transform table $T(-s) = -\rho^3$ so that

$$\frac{1}{\rho^2}T(v) - v(0) - \rho v'(0) + (\lambda_1+\lambda_2)\left[\frac{T(\rho)}{\rho} - \rho v(0)\right] - \lambda_1\lambda_2\, T(\rho) = -\rho^3 \text{ and}$$

$$T(\rho)\left[\frac{1}{\rho^2} + \frac{(\lambda_1+\lambda_2)}{\rho} - \lambda_1\lambda_2\right] = -\rho^3 + 0 + p + 0\ .$$

Thus

$$T(p)\left[\frac{1+p(\lambda_1+\lambda_2)-\lambda_1\lambda_2 p^2}{p^2}\right] = p - p^3$$

$$T(p)[1 + p(\lambda_1+\lambda_2) - \lambda_1\lambda_2\, p^2] = p^2(p - p^3).$$

Resolving to partial fraction

$$T(p) = p^2\left[\frac{p-p^3}{1+(\lambda_1+\lambda_2)p-\lambda_1\lambda_2 p^2}\right] = p^2\left[\frac{p-p^3}{(1-\gamma p)(1-\beta p)}\right]$$

$$\therefore \left[\frac{p-p^3}{(1-\gamma p)(1-\beta p)}\right] = \frac{A}{(1-\gamma p)} + \frac{B}{(1-\beta p)}$$

or

$$p - p^3 = A(1-\beta p) + B(1-\gamma p).$$

Let $p = {}^{1}/_{\beta}$, then $B = \frac{\beta^2-1}{\beta^2(\beta-\gamma)}$. Also for $p = \frac{1}{\gamma}$, $A = \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}$.

Therefore

$$\frac{p-p^3}{(1-\gamma p)(1-\beta p)} = \frac{A}{(1-\gamma p)} + \frac{B}{(1-\beta p)} = \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)(1-\gamma p)} + \frac{\beta^2-1}{\beta^2(\beta-\gamma)(1=\beta p)}\,,$$

and

$$T(p) = p^2\left[\frac{p-p^3}{(1-\gamma p)(1-\beta p)}\right] = p^2\left[\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)(1-\gamma p)}\right] + p^2\left[\frac{\beta^2-1}{\beta^2(\beta-\gamma)(1-\beta p)}\right]$$

$$\Longrightarrow \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}\cdot\frac{\rho^2}{(1-\gamma\rho)} + \frac{\beta^2-1}{\beta^2(\beta-\gamma)}\cdot\frac{\rho^2}{(1-\beta\rho)},$$

$$\Longrightarrow v(s) = \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}e^{\gamma s} + \frac{\beta^2-1}{\beta^2(\beta-\gamma)}e^{\beta s}.$$

Mohand transform

For this transform, taking the Mohand transform of the equation

$$v^2R(v) - v^3 f(0) - v^2 f'(0) + (\lambda_1+\lambda_2)[vR(v) - v^2 f(0)] - \lambda_1\lambda_2\, R(v) = R(-s)$$

Using p in place of v and f for v

$$p^2R(\rho) - p^3v(0) - p^2v'(0) + (\lambda_1+\lambda_2)[pR(p) - p^2v(0)] - \lambda_1\lambda_2\, R(p) = -1$$

from the inverse Mohand table.

$$i.e.\, R(-s) = -1$$

$$R(p)[p^2 + p(\lambda_1+\lambda_2) - \lambda_1\lambda_2] = -1 + p^3v(0) + p^2v'(0) + (\lambda_1+\lambda_2)p^2v(0)$$

$$R(p)[p^2 + p(\lambda_1+\lambda_2) - \lambda_1\lambda_2] = -1 + p^3(0) + p^2(1) + (\lambda_1+\lambda_2)p^2(0)$$

$$R(p)[p^2 + p(\lambda_1+\lambda_2) - \lambda_1\lambda_2] = -1 + p^2$$

$$R(p)[p^2 + p(\lambda_1+\lambda_2) - \lambda_1\lambda_2] = p^2 - 1$$

Knowing that $\lambda_1+\lambda_2 = \gamma + \beta$ & $\lambda_1\lambda_2 = \gamma\beta$

$R(p) = \frac{p^2-1}{p^2+p(\lambda_1+\lambda_2)-\lambda_1\lambda_2} = \frac{p^2-1}{(p-\gamma)(p-\beta)}$

In order to achieve our aim we will multiply the numerator and denominator by p^2.

i.e. $\frac{p^2(p^2-1)}{p^2(p-\gamma)(p-\beta)}\ p^2\left[\frac{A}{p^2}+\frac{B}{(p-\gamma)}+\frac{C}{p-\beta}\right]$

Resolving to partial fraction

$p^2-1 = A(p-\gamma)(p-\beta)+B(p-\beta)p^2+Cp^2(p-\gamma)$ Let $p=0$ then $-1 = A(\gamma\beta)$, $A = \frac{-1}{\gamma\beta}$. For, $let\ p=\gamma$ then $B = \frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}$ and for $p=\beta, C = \frac{\beta^2-1}{\beta^2(\beta-\gamma)}$.

$$\therefore\ p^2\left[\frac{A}{p^2}+\frac{B}{p-\gamma}+\frac{C}{p-\beta}\right] = p^2\left[\frac{-1}{\gamma\beta p^2}+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)(p-\gamma)}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)(p-\beta)}\right]$$
$$= \left[\frac{-p^2}{\gamma\beta p^2}+\frac{(\gamma^2-1)p^2}{\gamma^2(\gamma-\beta)(p-\gamma)}+\frac{(\beta^2-1)p^2}{\beta^2(\beta-\gamma)(p-\beta)}\right].$$

Thus

$$v(s) = \frac{-1}{\gamma\beta}+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}e^{\gamma s}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)}e^{\beta s}$$

From the inverse Mohand table $R(1) = s$.

Recall also that $\gamma\beta = -s$, thus

$$v(s) = 1+\frac{\gamma^2-1}{\gamma^2(\gamma-\beta)}e^{\gamma s}+\frac{\beta^2-1}{\beta^2(\beta-\gamma)}e^{\beta s}\ .$$

Conclusion

The definition and analysis of the new integral transforms (Aboodh transform, Elzaki transform, Mohand transform) to the solution of the second order linear Black Schole's pde has been demonstrated. The three transforms are well connected but the Elzaki transform showed a little distinction from others.

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