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Exponentiated Gumbel Family of Distributions; Properties and Applications

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Abstract

In this study, we proposed a family of distribution called the Exponentiated Gumbel family of distributions. Its density function is symmetric, right-skewed, left-skewed and reverse-J shaped with increasing, decreasing and inverted bathtub hazard rate function. Some special members of the model were obtained. Mathematical properties of the proposed family of distributions derived include quantile, generating functions, order statistics, and Renyi's entropy. Bivariate extension of the proposed family was discussed. The maximum likelihood method was employed in obtaining the parameter estimates of the Exponentiated Gumbel family. The usefulness of the proposed family was illustrated using two real datasets

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Introduction

Many families or classes of univariate distributions have been proposed in the literature by adding parameters to the baseline distribution. The addition of extra parameters has greatly improved the flexibility and goodness of fit of the generated class. Some known generated class of distributions include: beta-G by Eugene *et al*. (2002), exponentiated type distributions by Nadarajah and Kotz (2006), gamma–G by Zografos and Balakrishnan (2009), Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G by Alexander *et al*. (2012), exponentiated generalized class by Cordeiro *et al*. (2013) , Transformed-Transformer (T-X) family by Alzaatreh *et al*. (2013), T-X{Y}-method based on quantile function by Aljarrah *et al.* (2014), T-R{Y}-approach which redefined T-X{Y} by Alzaatreh *et al*. (2014) , Kumaraswamy Marshall-Olkin family Alizadeh *et al*. (2014) , exponentiated Weibull generated family by Hassan and Elgarhy (2016), Lindley-G by Cakmakyapan and Ozel (2016) and extended Marshal-Olkin Gumbel –G by Ugwuowo and Nwezza (2018).

Priliminaries

Given a random variable X with probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$. Let $r(t)$ be the pdf of a continuous random variable $T \in [a,b]$. Alzaatreh *et al*. (2013) defined the cdf $G(x)$ of a new family of distributions of a random variable X as

$$G(x) = \int_a^{W(F(x))} r(t)\,dt = R\big(W(F(x))\big) \tag{2.1}$$

Where $W(F(x))$ satisfies the following conditions; $W(F(x)) \in [a,b]$

$W(F(x))$ is differentiable and monotonically non-decreasing

$W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$.

In this paper, a new family of distribution called Exponentiated Gumbel family $(EGu-G)$ of distributions is proposed. We used exponentiated Gumbel distribution as a generator. The *pdf* and *cdf* of exponentiated Gumbel as given by Nadarajah (2006) respectively are

$$r(t)=\frac{\alpha}{\sigma}\left[1-\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\exp\left(-\frac{t-\mu}{\sigma}\right) \quad (2.2)$$

and

$$R(t)=1-\left[1-\exp\left\{-\exp\left(-\frac{t-\mu}{\sigma}\right)\right\}\right]^{\alpha} \quad (2.3)$$

$$-\infty<t<\infty,-\infty<\mu<\infty,\sigma>0,\alpha>0$$

This paper aims to propose a family of distributions using exponentiated Gumbel distribution as a generator. The objectives are: to generate a flexible family of continuous distributions; obtain skewed distributions from symmetric ones; generate models with various hazard rate function (hrf) shapes; generate J-shaped, reverse J shaped, symmetric, right-skewed or left-skewed distributions; providing models that produce better fits than other distributions with the same baseline distribution.

Taking $W(F(x,\xi))$ as the logit of *cdf* of a distribution, thus

$$W(F(x;\xi))=\log\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)$$

where ξ is a vector of parameters in the baseline distribution. From (2.1) we have

$$G(x)=\int_a^{\log\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)} r(t)\,dt=R\left(\log\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)\right) \quad (2.4)$$

(2.3) and (2.2) can be written as

$$R(t)=1-\left[1-\exp\left\{-B\exp\left(-\frac{t}{\sigma}\right)\right\}\right]^{\alpha} \quad \text{and}$$

$$r(t)=\frac{\alpha}{\sigma}\left[1-\exp\left\{-B\exp\left(-\frac{t}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-B\exp\left(-\frac{t}{\sigma}\right)\right\}B\exp\left(-\frac{t}{\sigma}\right)$$

respectively; where $B=\exp\left(\frac{\mu}{\sigma}\right)$. From (2.4) we have

$$G(x)=1-\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha} \quad (2.5)$$

Differentiating equation (2.5) and simplifying gives

$$g(x)=\frac{\alpha Bf(x;\xi)\left(F(x;\xi)\right)^{-\frac{1}{\sigma}-1}}{\sigma\left(1-F(x;\xi)\right)^{-\left(\frac{1}{\sigma}-1\right)}}\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1} \quad (2.6)$$

If $\alpha=1$, equation (2.5) and (2.6) reduces to the *cdf* and *pdf* of Gumbel-X distribution defined by Al-Aqtash *et al*. (2015). The hazard rate function (hrf) and survival function (sf) of $EGu-G$ respectively are given by

$$h(x)=\frac{\alpha Bf(x;\xi)\left(F(x;\xi)\right)^{-\left(\frac{1}{\sigma}+1\right)}\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}}{\sigma\left(1-F(x;\xi)\right)^{-\left(\frac{1}{\sigma}-1\right)}\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]} \quad (2.7)$$

and

$$S(x)=\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{1-F(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}$$

The rest of the paper is organized as follows. In section three, we present newly generated distributions from the proposed family. Mathematical properties including the shape of the density function, moments, moment generating function, entropy and order statistics of the proposed family are presented in section 4. Section 5 discussed the bivariate extension of the proposed family. Estimation of the parameters of the proposed family is presented in section 6. Two real datasets applications to illustrate the flexibility of some of the members of the proposed family are shown in section 7. The paper is concluded in section 8.

Special Models

In this section, some special models generated with the cdf in (5) are presented.

Exponentiated Gumbel-Normal

Letting $F(x;\xi)=\Phi(Z)$ in (2.5). The cdf and pdf of Exponentiated Gumbel-Normal$(EGuN)$ distribution respectively are given by

$$G_N(x)=1-\left[1-\exp\left\{-B\left(\frac{\Phi(z)}{1-\Phi(z)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}$$

and

$$g_N(x)=\frac{\alpha B\phi(z)\left(\Phi(z)\right)^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma\left(1-\Phi(z)\right)^{-\left(\frac{1}{\sigma}-1\right)}}\exp\left\{-B\left(\frac{\Phi(z)}{1-\Phi(z)}\right)^{-\frac{1}{\sigma}}\right\}\left[1-\exp\left\{-B\left(\frac{\Phi(z)}{1-\Phi(z)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}$$

where $\phi(.)$ and $\Phi(.)$ are the pdf and cdf of the standard normal distribution. Figure 1 is some plots of the pdf and hrf of $EGuN$ for selected parameter values. Figure 1 shows that $EGuN$ pdf can be right-skewed, left-skewed and unimodal while the hrf is increasing and J-shaped.

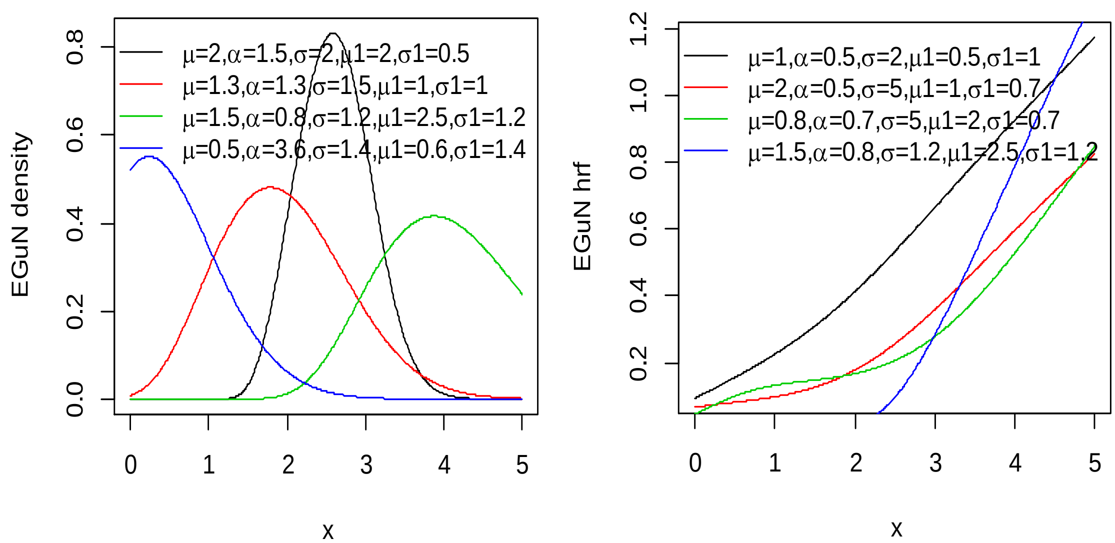

Figure 1. Plots of $EGuN$ pdf (left) and hrf (right) for selected parameter values

Exponentiated Gumbel-Weibull

Let the baseline distribution in (2.5) be a Weibull distribution with cdf $F(x;\xi)=1-\exp\left[-\left(\frac{x}{b}\right)^{a}\right]$ and pdf $f(x;\xi)=\frac{a}{b}\left(\frac{x}{b}\right)^{a-1}\exp\left[-\left(\frac{x}{b}\right)^{a}\right]$, $\xi=(a,b)$. The pdf and cdf of exponentiated Gumbel Weibull $(EGuW)$ is obtained directly from (2.5) and (2.6) by substituting for the pdf and cdf of Weibull distribution as defined above. Thus we have

$$G_W\left(x\right)=1-\left[1-\exp\left\{-B\left(-\frac{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}{\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}$$

$$g_w\left(x\right)=\frac{\alpha Ba\left(\frac{x}{b}\right)^{a-1}}{\sigma b\left(1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)\right)^{\left(\frac{1}{\sigma}+1\right)}}\exp\left(\left(-\left(\frac{1}{\sigma}\left(\frac{x}{b}\right)^{a}+B\left(-\frac{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}{\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}\right)^{-\frac{1}{\sigma}}\right)\right)\right)$$

$$\times\left[1-\exp\left\{-B\left(\frac{1-\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}{\exp\left(-\left(\frac{x}{b}\right)^{a}\right)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}$$

Figure 2 shows the plots of $EGuW$ distribution's pdf and hrf for selected parameter values. Figure 2 reveals that the density $EGuW$ can be right-skewed, reverse J-shaped, and unimodal, while the hrf is increasing, decreasing and unimodal.

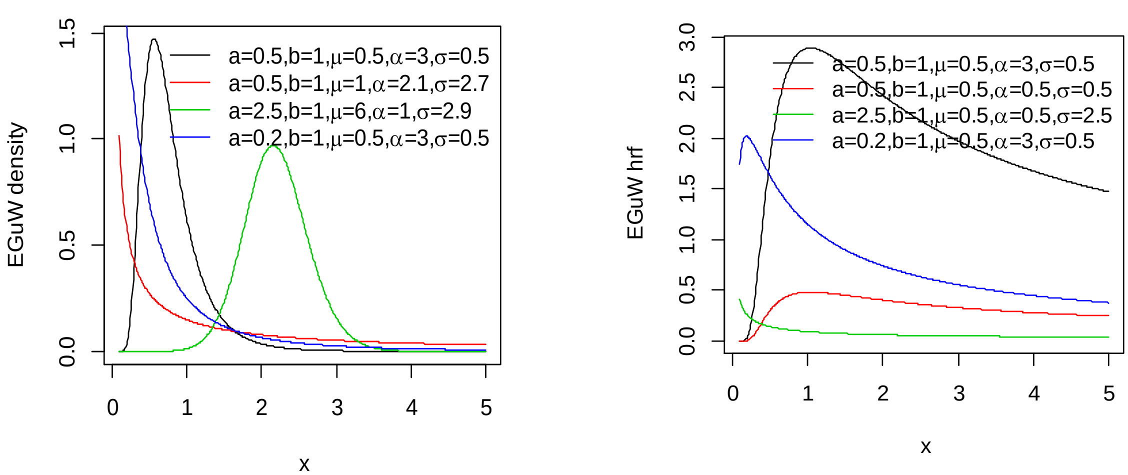

Figure 2. Plots of $EGu-W$ pdf (left) and hrf (right) for selected parameter values

3.3 Exponentiated Gumbel-Lomax

The pdf and cdf of Lomax distribution for $x>0$, $a>0$ $b>0$, respectively are given by $f(x;\xi)=\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}$ and $F(x;\xi)=1-\left(1+\frac{x}{b}\right)^{-a}$. Using (2.5) we obtain the cdf of exponentiated Gumbel Lomax $(EGuL)$ as

$$G_L(x)=1-\left[1-\exp\left\{-B\left(\frac{1-\left(1+\frac{x}{b}\right)^{-a}}{\left(1+\frac{x}{b}\right)^{-a}}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}$$

and using (2.6), the pdf of $EGuL$ is given by

$$g_L(x)=\frac{\alpha Ba\left(1+\frac{x}{b}\right)^{-\left(\frac{a}{\sigma}+1\right)}}{\sigma b\left(1-\left(1+\frac{x}{b}\right)^{-a}\right)^{\left(\frac{1}{\sigma}+1\right)}}\exp\left\{-B\left(\frac{1-\left(1+\frac{x}{b}\right)^{-a}}{\left(1+\frac{x}{b}\right)^{-a}}\right)^{-\frac{1}{\sigma}}\right\}\left[1-\exp\left\{-B\left(\frac{1-\left(1+\frac{x}{b}\right)^{-a}}{\left(1+\frac{x}{b}\right)^{-a}}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}$$

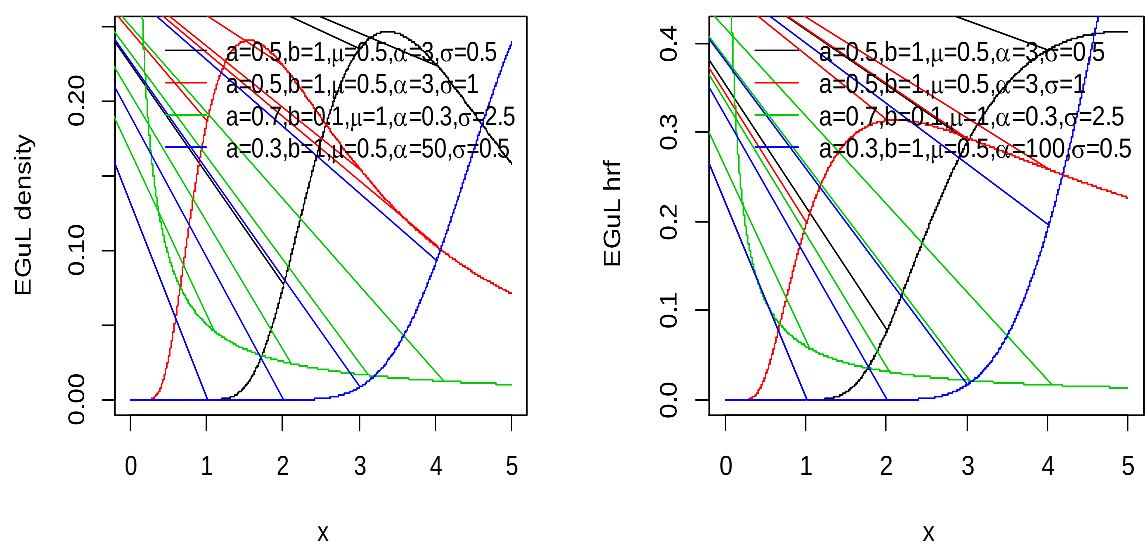

Figure 3. Plots of selected parameter values of $EGuL$'s pdf (left) and hrf (right)

Figure 3 is the plots of $EGuL$ distribution's pdf and hrf for selected parameter values. It reveals that $EGuL$ density can be unimodal reverse J-shaped, and increasing, while the hrf may be J-shaped unimodal and decreasing.

MATHEMATICAL PROPERTIES

The shapes of the density function of $EGuG$ can be described analytically. The critical points of the density function are the roots of the equation given below

$$\frac{d\log\left[g(x)\right]}{dx}=\frac{f'(x;\xi)}{f(x;\xi)}-f(x;\xi)\left[\frac{\left(\frac{1}{\sigma}+1\right)}{F(x;\xi)}+\frac{\left(\frac{1}{\sigma}-1\right)}{\left(1-F(x;\xi)\right)}\right]+\frac{Bf(x;\xi)\left(F(x;\xi)\right)^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma\left(1-F(x;\xi)\right)^{-\left(\frac{1}{\sigma}-1\right)}}\times\left[1-\frac{(\alpha-1)\exp\left\{-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right\}\right]}\right] \quad (4.1)$$

There may be more than one root to (4.1). If $x=x_0$ is a root of (4.1), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether $\psi(x_0)<0,\ \psi(x_0)>0$ or $\psi(x_0)=0$ where $\psi(x_0)=\frac{d^2\log\left(g(x)\right)}{dx^2}$

The shape of the hrf can be determined by taking the log of (2.7) and differentiating with respect to x and equating it to zero. The critical points of hrf are the roots of (4.2).

The roots of (4.2) may be more than one. If $x=x_0$ is a root of (4.2), then it corresponds to a local maximum, local minimum or a point of inflexion depending on whether $\zeta(x_0)<0,\zeta(x_0)>0$ or $\zeta(x_0)=0$,

$$\frac{d\log\left[h(x)\right]}{dx}=\frac{f'(x;\xi)}{f(x;\xi)}-f(x;\xi)\left[\frac{\left(\frac{1}{\sigma}+1\right)}{F(x;\xi)}-\frac{\left(\frac{1}{\sigma}-1\right)}{\left(1-F(x;\xi)\right)}\right]+\frac{Bf(x;\xi)\left(F(x;\xi)\right)^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma\left(1-F(x;\xi)\right)^{-\left(\frac{1}{\sigma}-1\right)}}\times\left[1-\frac{\exp\left\{-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right\}}{\left[1-\exp\left\{-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right\}\right]}\right] \quad (4.2)$$

where $\zeta(x_0)=\frac{d^2\log\left(h(x)\right)}{dx^2}$

Quantile Function of $EGu-G$ Family

The quantile function of $EGu-G$ family is obtained by inverting the cdf of the $EGu-G$ family as given in (2.5). The quantile function of $EGu-G$ is given by

$$X = F^{-1}\left[\left(1+\left\{-\frac{1}{B}\log\left[1-(1-u)^{\frac{1}{\alpha}}\right]\right\}^{\sigma}\right)^{-1}\right] = Q(u) \qquad (4.3)$$

The median of the EG-G family is given by $Q\left(\frac{1}{2}\right)$

$$Q\left(\frac{1}{2}\right) = F^{-1}\left[\left(1+\left\{-\frac{1}{B}\log\left[1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right]\right\}^{\sigma}\right)^{-1}\right]$$

The effects of the parameters of the $EGu-G$ on the skewness S and kurtosis K can be examined using quantile measures. Skewness and kurtosis are used to measure the degree of long tail and the degree of tail heaviness respectively. Skewness and kurtosis are calculated respectively using the relationships of Galton (1983) and Moor (1988). The Galton's and Moors's measures of skewness and kurtosis exist for distributions without moment and are less sensitive to outliers. Using the quantile function in (4.3), the Galton's skewness and Moors kurtosis of the proposed family is given by

$$S = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)}$$

and

$$K = \frac{Q\left(7/8\right) - Q\left(5/8\right) + Q\left(3/8\right) - Q\left(1/8\right)}{Q\left(6/8\right) - Q\left(2/8\right)}$$

A 3-dimensional plot of the Galton's skewness and Moors' kurtosis against α and μ for $\sigma = 0.5,\ \mu_1 = 0$ and $\sigma_1 = 1$ of the $EGuN$ distribution are presented in Figure 4. Figure 4 reveals that as α increases for (fixed μ) the skewness and kurtosis decreases. Hence we can conclude that the parameter α has more effect on the skewness and kurtosis than the parameter μ .

Theorem: The quantile function of the $T-X$ family defined in (2.4) is given by

$$Q_{T-X}(u) = F^{-1}\left(\left(\frac{\exp\left(R^{-1}(u)\right)}{1+\exp\left(R^{-1}(u)\right)}\right);\xi\right), \quad 0<u<1$$

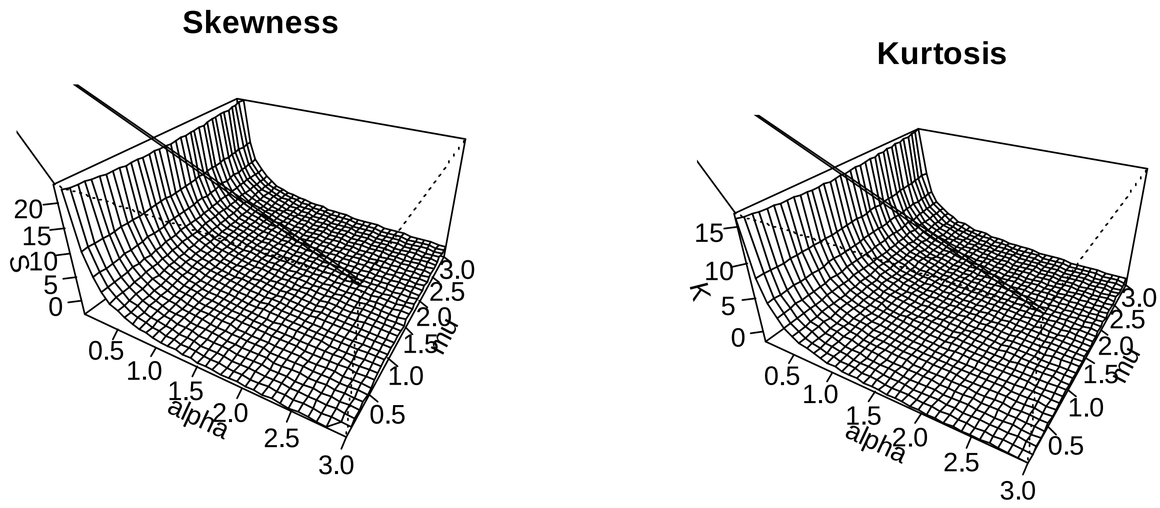

Figure 4. Galton's skewness (S) and Moore's kurtosis (K) for $EGuN$'s distribution. $(\sigma = 0.5, \mu_1 = 0, \sigma_1 = 1)$.

Where $F^{-1}(.)$ is the quantile function of the random variable X with distribution function $F(x;\xi)$ and $R^{-1}(.)$ is the quantile function of the random variable T with distribution function $R(t)$.

Proof: The proof of this theorem follows by equating (4) to u and solving for x accordingly.

Useful expansions

The following expansions are very useful in obtaining a linear representation for $EGu-G$ family and derivation of some of its important properties such as the order statistics and entropy.

$$(1-z)^k = \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} z^j \qquad |z|<1 \tag{4.4}$$

$$\exp(-z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^j$$

(4.5)

Linear Representation of $EGu-G$ Density

Considering the expansions given in (4.4) and (4.5), (2.5), the cdf of $EGu-G$ may be represented as

$$G(x) = 1 + \sum_{q=0}^{\infty} w_q F(x,\xi)^q \tag{4.6}$$

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$$w_q = (-1)^q \sum_{j,k,m=0}^{\infty} \sum_{n=q}^{\infty} \frac{(-1)^{j+k+m+n+1}}{k!} (Bj)^k \binom{\alpha}{j} \binom{\frac{k}{\sigma}}{m} \binom{m-\frac{k}{\sigma}}{n} \binom{n}{q}$$

In literature if $F(x)$ is any arbitrary cdf of a random variable, then for $\theta > 0$. $G(x) = F(x)^{\theta}$ and $g(x) = \theta f(x) F(x)^{\theta-1}$ are the cdf and pdf of exponentiated-G distribution pioneered by Mudholkar and Srivastava (1993). Thus some of the mathematical properties of the proposed distribution can be obtained using the properties of the exponentiated-G family. Hence (4.5) can be written as

$$G(x) = 1 + \sum_{q=0}^{\infty} w_q H_q(x;\xi) \tag{4.7}$$

$H_q(x;\xi)$ is the cdf of exponentiated G distribution. By differentiating (4.7), the $EGu-G$ cdf reduces to

$$g(x) = \sum_{q=0}^{\infty} w_{q+1} h_{q+1}(x;\xi) \tag{4.8}$$

Where $h_q(x)$ is exponentiated-G density with power parameter q. (4.8) shows that $EGu-G$ can be expressed as a linear combination of exponentiated-G densities. (4.7) and (4.8) are the major results of this section.

Moments

Let Y_{q+1} be a random variable distributed as the baseline pdf with exponentiated-G distribution with power parameter $q+1$ and X, a random variable from $EGu-G$ family. Using equation (4.8) the rth noncentral moment of X can be derived using two formulae, firstly

$$E(X^r) = \sum_{q=0}^{\infty} w_{q+1} E(Y_{q+1}^r) \tag{4.9}$$

Nadarajah and Kotz (2006) obtained moments of some exponentiated-G distribution. These moments can be very useful in obtaining $E(X^r)$.

Secondly, the moments of $EGu-G$ can be obtained from (4.3) using the quantile function of the baseline distribution as

$$E(X^r) = \sum_{q=0}^{\infty} (q+1) w_{q+1} I(r,q) \tag{4.10}$$

Cordeiro and Nadarajah (2011) derived $I(r,q)$ for some distribution.

Where $I(r,q) = \int_{-\infty}^{\infty} x^r F(x;\xi)^q f(x;\xi)\,dx = \int_0^1 (Q(u))^r u^q du$

4.4 Moment Generating Function

Given that $M_X(t) = E(e^{tX})$ be the mgf of a random variable X from $EGu-G$. The mgf of X firstly is given by

$$M_X(t) = \sum_{q=0}^{\infty} w_{q+1} M_{Y_{q+1}}(t) \tag{4.11}$$

where $M_{Y_{q+1}}(t)$ is the mgf of Y_{q+1}. Thus $M_X(t)$ can be obtained from the exponentiated-G mgfs.

Secondly, the mgf of $EGu-G$ can be obtained from the pdf of $EGu-G$ as

$$M_X(t)=\sum_{q=0}^{\infty}(q+1)w_{q+1}I^*(t,q) \quad (4.12)$$

where $I^*(t,q)=\int_{-\infty}^{\infty} e^{tx}F(x;\xi)^q f(x;\xi)dx=\int_0^1 e^{tQ(u)}u^q du$

The mgfs of members of $EGu-G$ family can be derived using (4.11) and (4.12).

Entropy

The entropy of a random variable X with density function $f(x)$ is the measure of the variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. The Renyi entropy of a random variable with density function $f(x)$ is given as

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty} f^{\gamma}(x)dx\right\} \quad (4.13)$$

For $\gamma>0$ and $\gamma\neq 1$

The Renyi entropy for the $EGu-G$ family is obtained directly from (4.13) by replacing $f(x)$ in (4.13) with (2.6). Using the expansions (4.4) and (4.5), we obtained the Renyi entropy for the $EGu-G$ as

$$I_R(\gamma)=\frac{\gamma}{1-\gamma}\log\left(\frac{\alpha B}{\sigma}\right)+\frac{1}{1-\gamma}\log\left\{Z_{i,k,m}I(m,\sigma,\gamma,k)\right\} \quad (4.14)$$

where $Z_{i,k,m}=\frac{(-1)^{j+k+m}}{k!}\left(B(\gamma+j)\right)^k\binom{\gamma(\alpha-1)}{j}\binom{\left(\frac{1}{\sigma}(k+\gamma)-\gamma\right)}{m}$

and

$$I(m,\sigma,\gamma,k)=\int_0^{\infty} f^{\gamma}(x;\xi)F(x;\xi)^{m-\left[\frac{1}{\sigma}(k+\gamma)-\gamma\right]}dx$$

4.6 Order Statistics

Order statistics has many applications in statistical theory and practice. Let $X_1, X_2\cdots X_n$ be a random sample from $EGu-G$ distribution family. Suppose that $X_{i,n}$ denotes the ith order statistics. The pdf of the ith order statistics can be expressed as

$$g_{i;n}(x)=\frac{g(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}G(x)^{i+j-1}$$

$$g_{i;n}(x)=\frac{\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}}{B(i,n-i+1)}g(x)G(x)^{i+j-1}, \quad (4.15)$$

where $B(.,.)$ is a beta function. Using (2.6) and (2.5) and applying the useful expansions in Section 4.1. Let $z=i+j+1$, we have

$$g_{i;n}(x)=\frac{\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}}{B(i,n-i+1)}\sum_{r=0}^{\infty}d_{r+1}h_{r+1}(x) \quad (4.16)$$

where

$$d_{r+1}=\frac{\alpha}{\sigma(r+1)}(-1)^{r}\sum_{q,t,k=0}^{\infty}\sum_{p=0}^{z}\sum_{m=r}^{\infty}\frac{(-1)^{p+q+t+k+m}}{t!}\binom{z}{p}\binom{\alpha(p+1)-1}{q}$$
$$\binom{c-1}{k}\binom{k-(c-1)}{m}\binom{m}{r}B^{t+1}(q+1)^{t}$$

and $h_{r+1}(x)=(r+1)\,f(x;\xi)\,F(x;\xi)^{(r+1)-1}$

Hence the ith order statistics of $EGu-G$ can be expressed as the linear combination of exponentiated- G of the baseline distribution. The mathematical properties of the order statistics can be obtained using the corresponding properties of the exponentiated-G of the baseline distribution. This is the major result in this section.

Bivariate Extension

Let $F(x,y;\xi)$ be the cdf of a bivariate baseline continuous distribution. We introduce the bivariate extension of the proposed model. The joint cdf $EGu-G$ is given by

$$G_{XY}(x,y)=1-\left[1-\exp\left\{-B\left(\frac{F(x,y;\xi)}{1-F(x,y;\xi)}\right)^{-1/\sigma}\right\}\right]^{\alpha} \quad (5.1)$$

The marginal cdf s are given by

$$G_{X}(x)=1-\left[1-\exp\left\{-B\left(\frac{F_1(x;\xi)}{1-F_1(x;\xi)}\right)^{-1/\sigma}\right\}\right]^{\alpha}$$

$$G_{Y}(y)=1-\left[1-\exp\left\{-B\left(\frac{F_2(y;\xi)}{1-F_2(y;\xi)}\right)^{-1/\sigma}\right\}\right]^{\alpha}$$

where $F_1(x;\xi)$ and $F_2(y;\xi)$ are the marginal cdf 's of $F(x,y;\xi)$. The joint pdf of X and Y can be obtained easily by $g_{X,Y}(x,y)=\frac{\partial^2 G_{X,Y}(x,y)}{\partial x\partial y}$

$$g_{X,Y}(x,y)=A(x,y)\frac{\alpha B}{\sigma}\frac{F(x,y;\xi)^{-(1/\sigma+1)}}{\bar{F}(x,y;\xi)^{-(1/\sigma-1)}}exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-1/\sigma}\right\}\left(1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-1/\sigma}\right\}\right)^{\alpha-1}$$

where $\bar{F}(x,y;\xi)=1-F(x,y;\xi)$ and

$$A(x,y)=f(x,y;\xi)-\frac{B(\alpha-1)}{\sigma}\frac{F(x,y;\xi)^{-\left(\frac{1}{\sigma}+1\right)}}{\bar{F}(x,y;\xi)^{-\left(\frac{1}{\sigma}-1\right)}}\frac{\partial F(x,y;\xi)}{\partial y}\frac{\partial F(x,y;\xi)}{\partial x}\left(1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right)^{-1}$$
$$+\frac{1}{F(x,y;\xi)\bar{F}(x,y;\xi)}\left[\frac{B}{\sigma}\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-\frac{1}{\sigma}}-\left(\frac{1}{\sigma}-1\right)\right]\frac{\partial F(x,y;\xi)}{\partial y}\frac{\partial F(x,y;\xi)}{\partial x}$$

The marginal pdf 's are given by

$$g_X(x)=\frac{\alpha B}{\sigma}\frac{F_1(x;\xi)^{-\left(\frac{1}{\sigma}+1\right)}f_1(x;\xi)}{\bar{F}_1(x;\xi)^{-\left(\frac{1}{\sigma}-1\right)}}exp\left\{-B\left(\frac{F_1(x;\xi)}{\bar{F}_1(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\left(1-exp\left\{-B\left(\frac{F_1(x;\xi)}{\bar{F}_1(x;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right)^{\alpha-1}$$

and

$$g_Y(y)=\frac{\alpha B}{\sigma}\frac{F_2(y;\xi)^{-\left(\frac{1}{\sigma}+1\right)}f_2(y;\xi)}{\bar{F}_2(y;\xi)^{-\left(\frac{1}{\sigma}-1\right)}}exp\left\{-B\left(\frac{F_2(y;\xi)}{\bar{F}_2(y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\left(1-exp\left\{-B\left(\frac{F_2(y;\xi)}{\bar{F}_2(y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right)^{\alpha-1}$$

The conditional cdf 's are

$$G_{X/Y}(x/y)=\frac{1-\left[1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}}{1-\left[1-exp\left\{-B\left(\frac{F_2(y;\xi)}{\bar{F}_2(y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}}$$

$$G_{Y/X}(y/x)=\frac{1-\left[1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}}{1-\left[1-exp\left\{-B\left(\frac{F_1(x,\xi)}{\bar{F}_1(x,\xi)}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha}}$$

The conditional pdf 's are

$$g_{X/Y}(x/y)=\frac{A(x,y)F(x,y,\xi)^{-\left(\frac{1}{\sigma}+1\right)}\bar{F}_2(y,\xi)^{-\left(\frac{1}{\sigma}-1\right)}}{\bar{F}(x,y,\xi)^{-\left(\frac{1}{\sigma}-1\right)}F_2(y;\xi)^{-\left(\frac{1}{\sigma}+1\right)}f_2(y;\xi)}\frac{exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}}{exp\left\{-B\left(\frac{F_2(y;\xi)}{\bar{F}_2(y;\xi)}\right)^{-\frac{1}{\sigma}}\right\}}$$

$$\times\frac{\left(1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-1/\sigma}\right\}\right)^{\alpha-1}}{\left(1-exp\left\{-B\left(\frac{F_2(y;\xi)}{\bar{F}_2(y;\xi)}\right)^{-1/\sigma}\right\}\right)^{\alpha-1}}$$

$$g_{Y/X}(y/x)=\frac{A(x,y)F(x,y;\xi)^{-(1/\sigma+1)}\bar{F}_1(x;\xi)^{-(1/\sigma-1)}}{\bar{F}(x,y;\xi)^{-(1/\sigma-1)}F_1(x;\xi)^{-(1/\sigma+1)}f_1(x;\xi)}\frac{exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-1/\sigma}\right\}}{exp\left\{-B\left(\frac{F_1(x;\xi)}{\bar{F}_1(x;\xi)}\right)^{-1/\sigma}\right\}}$$

$$\times\frac{\left(1-exp\left\{-B\left(\frac{F(x,y;\xi)}{\bar{F}(x,y;\xi)}\right)^{-1/\sigma}\right\}\right)^{\alpha-1}}{\left(1-exp\left\{-B\left(\frac{F_1(x;\xi)}{\bar{F}_1(x;\xi)}\right)^{-1/\sigma}\right\}\right)^{\alpha-1}}$$

Estimation

Here we obtain maximum likelihood estimates (MLEs) of the model parameters of the proposed family. Let $x_1, x_2 ..... x_n$ be observed samples from $EGu-G$ distribution with parameters μ,σ,α and ξ .Letting $\Theta=(\mu,\sigma,\alpha,\xi)^T$ be the $r\times 1$ unknown parameter vector. The log likelihood function for the vector of parameters is given by $l=\log\left[\prod_{i=1}^{n}\left(g(x;\Theta)\right)\right]$

$$l=n\log\alpha+n\frac{\mu}{\sigma}-n\log\sigma+\sum_{i=1}^{n}\log f(x;\xi)-\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\log F(x;\xi)+\left(\frac{1}{\sigma}-1\right)\sum_{i=1}^{n}\left(1-F(x;\xi)\right)$$
$$+\sum_{i=1}^{n}\left(-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right)+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\left(-B\left(\frac{F(x;\xi)}{\left(1-F(x;\xi)\right)}\right)^{-\frac{1}{\sigma}}\right)\right] \qquad (6.1)$$

The logliklihood function can be maximized directly by using Adequacy Model or fitdistrplus packages in R or by differentiating (6.1) partially with respect to μ,σ,α and ξ and solving the resulting nonlinear equations.

Applications

Here, two real data sets are used to fit some special models from the exponentiated Gumbel-G family; Exponentiated Gumbel Lomax $(EGuL)$ and Exponentiated Gumbel Weibull $(EGuW)$. To prove empirically the potentiality of these models, their fits are compared with fits of other competitive models. In each case, the parameters in the model are estimated using the Maximum likelihood method (ML) using fitdistrplus package in R statistical software.

The $EGuL$ is compared with Lomax (L), Exponentiated Lomax (EL), Exponentiated Generalized Lomax (EGL), Kumaraswamy Lomax (KL), Exponentiated Kumaraswamy Lomax (EKL), and Beta Lomax (BL), while $EGuW$ is compared with Weibull (W), Exponentiated Weibull (EW), Exponentiated Generalized Weibull (EGW), Kumaraswamy Weibull (KW), Exponentiated Kumaraswamy Weibull (EKW) and Beta Weibull (BW). The Anderson-Darling (A^*) and Cramer-von Mises (W^*) statistics are used in the comparison of $EGuL$ and $EGuW$ with other models. These two statistics are widely used in the comparison of non-nested models. In general, the smaller the values of these statistics, the better the fit of the distribution to the data.

Example 1

The first dataset corresponds to the breaking stress of carbon fibers (in Gba) from Nichols and Padgett (2006). We fitted $EGuL$ and other competing models to this dataset. The MLEs of the parameters and their standard errors (in parentheses) are listed in Table 1a while the values of the A^* and W^* are listed in Table 1b. Table 1b reveals $EGuL$ has the smallest values of A^* and W^* among of the fitted models. Hence the $EGuL$ is the best among the models fitted to the data. The histogram of the dataset, estimated pdf and cdf are shown in Figure 5.

Table 1a: MLEs of parameters (standard errors in parenthesis)

Distributions	Estimates				
$EGuL(a,b,\mu,\alpha,\sigma)$	13.1787 (43.0041)	7.63 (27.69)	8.45 (8.77)	18.17 (65.77)	3.91 (3.25)
$L(a,b)$	4071646.49 (11863.28)	10674210 (29.75)			
$EL(a,b,\theta)$	3104301.70 (11894.45)	3079190 (25.40)	7.71 (0.77)		
$EGL(a,b,\theta,\alpha)$	44792.16 (23134.72)	281539.99 (76.1010)	6.37 (3.34)	7.79 (1.50)	
$KL(a,b,\theta,\alpha)$	5.71 (23.39)	43.49 (165.97)	3.44 (0.75)	56.51 (147.60)	
$EKL(a,b,\theta,\alpha,\beta)$	3.14 (2.26)	6.00 (6.99)	9.42 (13.24)	13.24 (10.08)	0.51 (0.57)
$BL(a,b,\alpha,\beta)$	6.13 (0.91)	6.78 (7.22)	2285.25 (1979.65)	8817.24 (696.58)	

Table 1b: Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics

	$EGuL$	L	EL	EGL	KL	EKL	BL
W^*	0.067	3.433	0.226	0.230	0.071	0.092	0.154
A^*	0.389	17.300	1.218	1.227	0.414	0.483	0.783

Example 2

The second dataset is on the exceedances of Wheaton river flood recently analyzed by Corderio *et al* (2018). We fitted $EGuW$ and other competing models to the dataset. Estimates of parameters of $EGuW$ and their standard errors are shown in Table 2a. Table 2b and Figure 6 shows that $EGuW$ provides the best fit when compared to other competing models in this application.

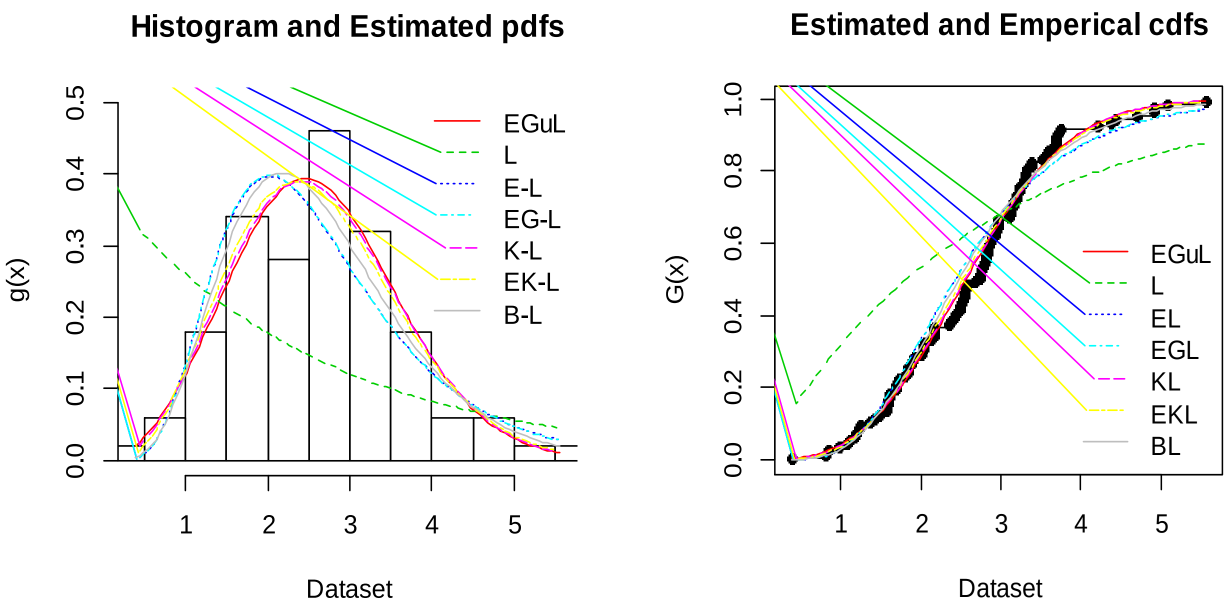


Figure 5. Plots of estimated pdf 's and cdf 's of $EGuL$ with other competing models and empirical cdf

Table 2a: MLEs of parameters (standard errors in parenthesis)

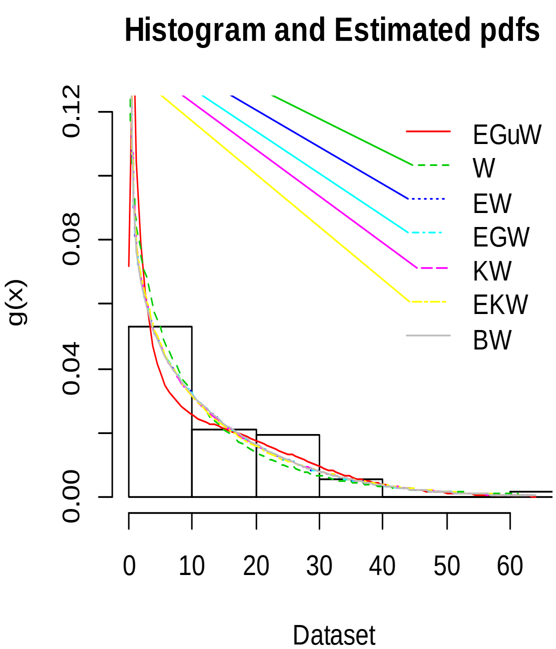
Distributions	Estimates				
$EGuW(a,b,\mu,\alpha,\sigma)$	1.57 (0.35)	7.45 (3.58)	2.05 (2.25)	0.58 (0.64)	2.75 (1.48)
$W(a,b)$	0.88 (0.08)	11.40 (1.60)			
$EW(a,b,\theta)$	1.37 (0.62)	19.92 (8.81)	0.51 (0.32)		
$EGW(a,b,\theta,\alpha)$	1.37 (0.62)	4.26 (43.59)	0.12 (1.69)	0.51 (0.32)	
$KW(a,b,\theta,\alpha)$	2.04 (2.08)	48.49 (60.00)	0.356 (0.37)	2.20 (2.66)	
$EKW(a,b,\theta,\alpha,\beta)$	1.51 (0.00)	5.89 (0.00)	1.45 (0.5225)	0.13 (0.03)	0.34 (0.09)
$BW(a,b,\alpha,\beta)$	1.29 (0.53)	10.77 (16.38)	0.54 (0.3055)	0.52 (0.74)	

Table 2b Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics

	$EGuW$	W	EW	EGW	KW	EKW	BW
W^*	0.037	0.168	0.125	0.125	0.116	0.120	0.123
A^*	0.246	0.945	0.752	.752	0.691	0.717	0.742

Conclusion

A new family of distribution called exponentiated Gumbel-G is proposed and studied in this paper. The $EGu-G$ has Gumbel –X proposed by Al-Aqtash *et al*. (2015) as a special case. The density of the proposed family was expressed as a linear combination of the exponentiated density of the baseline pdf . The $EGu-G$ family has the capacity to generate distributions whose hazard rate function is very flexible. We derived the quantile function, moments, entropy, order statistics and the bivariate extension of $EGu-G$ family. The estimation of the parameters of the proposed family was done using the method of maximum likelihood. Two datasets were used to illustrate the potentiality and usefulness of the proposed family.



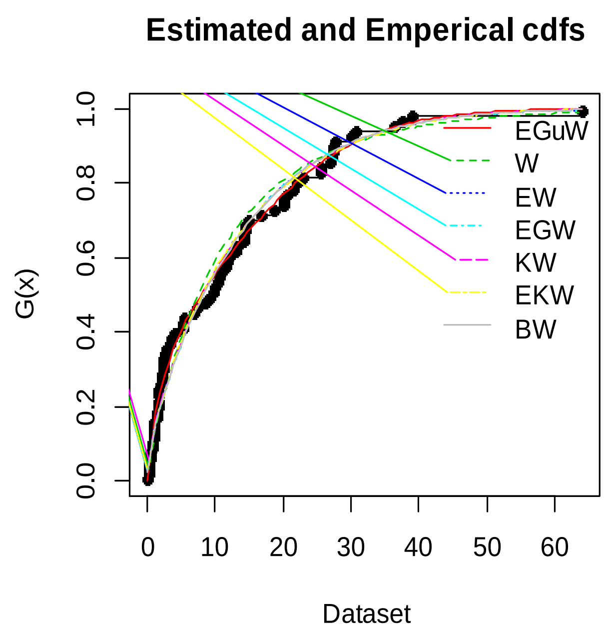


Figure 6. Plots of estimated pdf 's and cdf 's of $EGuW$ with other competing models and empirical cdf

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