Effect of death rate in determining the optimal investment strategies for defined contribution (DC) pension fund with multiple contributors

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Abstract: The effect of death rate in determining the optimal investment strategies for defined contribution (DC) pension fund with multiple contributors was investigated using a modified model. We assume a case where the wealth of death pensioners is not added to the pension wealth and also when their wealth is added to pension wealth. Using this model we obtained optimized problems for the two assumptions using Jacobi Hamilton equation and solve the problems using Legendre transform to obtain an explicit solution of the optimal investment strategies for CARA utility function.
We observed that the optimal investment strategies with the death pensioners' wealth is greater compared to one without their wealth. This model has shown that the wealth of the death pensioners has an effect on the overall investment strategies hence the pension manager makes more interest with the surplus from the death pensioners' wealth and loses more if the investment fails.

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Introduction.

In a pension fund system with multiple contributors, it is expected that payment are made to contributors who have retired from service and the payment continues till the death of a specific contributor after which payment is stopped for that particular contributor. The payment is a stochastic process and assumes the Brownian motion with drift (Dawei and Jingyi, 2014). Defined contribution pension scheme as explained by (Antolin *et al.*, 2010) is very crucial in retirement income system in most countries and it forms a decider of the old age income adequacy for future retirees. This system requires the need to understand the risks involved with the income provided by this plan and the need to solve the optimal investment strategies problem to determine how to invest and maximize profit and reduce the risk involve. The most commonly used utility functions are the constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA), (Cairns *et al.*, 2006) studied optimal dynamic asset allocation for defined contribution pension plans using CRRA) utility function (Gao, 2008), (Boulier *et al.*, 2001), (Deelstra *et al.*, 2003), (Xiao *et al.*, 2007) also used (CRRA) utility function to obtain the optimal investment strategies in a DC pension fund. (Battocchio and Menoncin, 2004), (Gao, 2009) obtained optimal investment strategy in a DC pension fund for a (CARA) utility function. The optimal investments for DC with stochastic interest rate have been studied in (Battocchio and Menoncin, 2004) where the interest rate was Vasicek model. (Chubing and Ximing, 2013), and (Gao, 2008) studied the affine interest rate which include the Cox- Ingeroll-Ross (CIR) model and Vasicek model. Recently, more attention has been given to constant elasticity of Variance (CEV) model in DC pension fund investment strategies. As Geometric Brownian motion (GBM) can be considered as a special case of the CEV model, such work extended the research in (Xiao *et al.*, 2007) where they applied the model to derive dual solution of a CRRA utility function via Legendre transform, also (Gao, 2009) extended the work in (Xiao *et al.*, 2007) by obtaining solutions for investor with CRRA and CARA utility function. (Dawei and Jingyi, 2014) improved on the work of (Gao, 2009) by modelling pension fund with multiple contributors where benefit payment are made after retirement. Explicit solution for CRRA and CARA using power transformation method was found. In this paper, we modify the model developed by (Dawei and Jingyi, 2014) and solve the optimized problem when the wealth of the death pensioners are added to the pension wealth and when is removed from the pension wealth via Legendre transformation and dual theory to obtain the explicit solution for CARA utility functions.

Mathematical Model

In a pension fund system with multiple contributors, it is expected that payment are remitted to contributors who have retired from service and the payment continues till the death of a specific contributor after which payment is stopped for that particular contributor. As stated by (Dawei and Jingyi, 2014), the payment is a stochastic process and assume the Brownian motion with drift as follows

$$dC(t) = adt - bdW^\circ(t), \quad (1)$$

where a and b are positive constants and denote the amount given to the retired contributors and that which is due death contributors which are out of the system. Assuming payment is made to the plan members after their death, model is given by

$$dC(t) = adt + bdW^\circ(t) \quad (2)$$

Assume the market is made up of risk free asset (cash and bond) and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{W^\circ(t), W_t(t)\}$ is a standard two dimensional Brownian motion. F is the filtration and denotes the information generated by the Brownian motion. Let $S_0(t)$ denote the price of the risk free asset, it dynamics is given by

$$\frac{dS_0(t)}{S_0(t)} = rdt. \quad (3)$$

Let $S_t(t)$ denote the risky asset and its dynamics is given based on its stochastic nature and the price process described by the CEV model as stated (Gao, 2009) is given by

$$\frac{dS_t(t)}{S_t(t)} = \mu dt + KS_t^{\beta} dW_t, \quad (4)$$

Where μ is an expected instantaneous rate of return of the risky asset and satisfies the general condition $\mu > r_0$. KS_t^{β} is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta < 0$.

Consider that in DC plans the contributions provided by the contributors are fixed and then without loss of generality, we assume that the number of contributors is constant and so is the contribution rate $c = (1+\theta)a$ with safety loading $\theta > 0$. If there is no investment, the dynamics of the surplus is given by

$$dR_1(t) = cdt - dC(t) = \theta adt + bdW^\circ(t) \quad (5)$$

$$dR_2(t) = cdt - dC(t) = \theta adt - bdW^\circ(t) \quad (6)$$

Let V_1 denote the wealth of pension fund at $t \in [0,T]$, associated with equation (5). Let π_1 denote the proportion of the pension fund invested in the risky asset S_t and $(1-\pi_1)$, the proportion invested in risk free asset. Hence the dynamics of the pension wealth is given by

$$dV_1 = \pi_1 V_1 \frac{dS_t(t)}{S_t(t)} + (1-\pi_1)V_1 \frac{dS_0(t)}{S_0(t)} + \theta a dt + bdW^{\circ}(t) \quad (7)$$

Substituting (3) and (4) into (7) we have;

$$dV_1 = [V_1(\pi_1(\mu - r) + r) + \theta a]dt + bdW^{\circ}(t) + \pi_1 V_1 K S_t{}^{\beta} dW_t. \quad (8)$$

Let V_2 denote the wealth of pension fund at $t \in [0,T]$, associated with equation (6) let π_2 denote the proportion of the pension fund invested in the risky asset S_t and$(1-\pi_2)$, the proportion invested in risk free asset. Hence the dynamics of the pension wealth is given by

$$dV_2 = \pi_2 V_2 \frac{dS_t(t)}{S_t(t)} + (1-\pi_2)V_2 \frac{dS_0(t)}{S_0(t)} + \theta a dt + bdW^{\circ}(t). \quad (9)$$

Substituting (3) and (4) into (9) we have

$$dV_2 = [V_2(\pi_2(\mu - r) + r) + \theta a]dt + bdW^{\circ}(t) + \pi_2 V_2 K S_t{}^{\beta} dW_t. \quad (10)$$

3. Optimization Problem

In this section we are interested in maximizing the utility of the plan contributor's terminal relative wealth. Let π_1 represent which is define to be the utility attained by the plan contributors from a given state v at time t as

$$H_{\pi_1}(t,s,v) = E_{\pi_1}\left[U\big(V(T)\big) \mid S(t) = s, V(t) = v\right] \quad (11)$$

where t is the time, s is the price of the risky asset and v is the wealth. Our aim is to obtain the optimal value function

$$H(t,s,v) = \sup_{\pi_1} H_{\pi_1}(t,s,v) \quad (12)$$

and the optimal strategy π_1 such that

$$H_{\pi_1}(t,s,v) = H(t,s,v). \quad (13)$$

The Jacobi Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem (8) is

$$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}b^2 H_{vv} + \sup\left\{\frac{1}{2}\pi^2 k^2 s^{2\beta} v^2 H_{vv} + \pi\left[(\mu - r)vH_v + k2s2\beta+1vHvs=0. \quad (14)$$

To obtain the first order maximizing condition for $\pi_1{}^*$, we differentiate the equation below with respect to π and solve for π

$$\frac{1}{2}\pi^2 k^2 s^{2\beta} v^2 H_{vv} + \pi\left[(\mu - r)vH_v + k^2 s^{2\beta+1} v H_{vs}\right] = 0, \quad (15)$$

such that

$$\pi_1{}^* = -\frac{\left[(\mu - r)H_v + k^2 s^{2\beta+1} H_{vs}\right]}{k^2 s^{2\beta} v H_{vv}}. \quad (16)$$

Substituting (16) into (14), we have

$$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}b^2 H_{vv} - \left[\frac{\left((\mu - r)H_v + k^2 s^{2\beta+1} H_{vs}\right)^2}{2k^2 s^{2\beta} H_{vv}}\right] = 0, \quad (17)$$

so that

$H_t + \mu s H_s + (rv + \theta a)H_v + \frac{1}{2}k^2 s^{2\beta+2}\Big[H_{ss} -$

$Hvs2Hvv+$

$$\frac{1}{2}b^2 H_{vv} - \frac{z^2(\mu-r)^2}{2k^2 s^{2\beta}}\frac{{H_v}^2}{H_{vv}} - (\mu - r)s\frac{H_v H_{vs}}{H_{vv}} = 0. \quad (18)$$

4. Legendre Transformation

Since the differential equation obtained in (18) is a non linear PDE and quite complex to solve, we will employ the Legendre transform and dual theory to transform it to a linear PDE

Theorem 4.1 : Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$L(z) = \max_x\{f(x) - zx\}, \quad (19)$$

where $L(z)$ is the Legendre dual of $f(x)$. (Jonsson and Sircar 2002).

Since $f(x)$ is convex, from theorem (4.1), we can defined the Legendre transform such that

$$\widehat{H}(t,s,z) = \sup\{H(t,s,v) - zv \mid 0 < x < \infty\}\ 0 < t < T, \quad (20)$$

where $\widehat{H}$ is the dual of H and $z > 0$ is the dual variable of v. The value of v where this optimum is attained is denoted by $g(t,s,z)$, so that

$$g(t,s,z) = \inf\left\{v \,\middle|\, \begin{matrix} H(t,s,v) \geq \\ zv + \widehat{H}(t,s,z) \end{matrix}\right\}\ 0 < t < T. \quad (21)$$

The function g and $\widehat{H}$are closely related and can be refers to the dual of H. These functions are related as follows

$$\widehat{H}(t,s,z) = H(t,s,g) - zg, \quad (22)$$

where $g(t,s,z) = v, H_v = z,\ g = -\widehat{H}_z$.

At terminal time, we denote

$\widehat{U}(z) = \sup\{U(v) - zv \mid 0 < v < \infty\}$

And

$G(z) = \sup\{v \mid U(v) \geq zv + \widehat{U}(z)$.

As a result

$$G(z) = (U')^{-1}(z). \quad (23)$$

Where G is the inverse of the marginal utility U and note that $H(T,s,v) = U(v)$

At terminal time T, we can define

$g(T,s,z) =$

$\inf_{v>0}\{\, v \mid U(v) \geq zv + \widehat{H}(t,s,z)\,\}\, and\, \widehat{H}(t,s,z) =$

$\sup_{v>0}\{\, U(v) - zv\}$

So that

$$g(T,s,z) = (U')^{-1}(z) \quad (24)$$

Next we differentiate (22) with respect to $t, s, and\ v$

$$H_t = \widehat{H}_t,\ H_s = \widehat{H}_s\,, H_v = z, H_{sv} = \frac{-\widehat{H}_{sz}}{\widehat{H}_{zz}}, H_{vv} = \frac{-1}{\widehat{H}_{zz}}\ ,$$

$$H_{sv} = \widehat{H}_{ss} - \frac{{\widehat{H}_{sz}}^2}{\widehat{H}_{zz}} \quad (25)$$

Substituting (25) into (18), we have

$\widehat{H}_t + \mu s\widehat{H}_s + (rv + \theta a)z + \frac{1}{2}k^2 s^{2\beta+2}\widehat{H}_{ss} -$

$\frac{1}{2}b^2\frac{1}{\widehat{H}_{zz}} - \frac{z^2(\mu-r)^2}{2k^2 s^{2\beta}}{\widehat{H}_{zz}}^2 - (\mu - r)sz\widehat{H}_{zz} = 0, (26)$

$${\pi_1}^* = -\frac{[(\mu-r)z\widehat{H}_{zz} - k^2 s^{2\beta+1}\widehat{H}_{sz}]}{k^2 s^{2\beta} v}. \quad (27)$$

Differentiating (26) and (27) with respect to z and using $v = g = -\widehat{H}_z$, we have

$g_t + rsg_s - (rg + \theta a) + \frac{1}{2}k^2 s^{2\beta+2}g_{ss} +$

$\left(\frac{(\mu-r)^2}{k^2 s^{2\beta}} - r\right)zg_z + \frac{1}{2}b^2\frac{g_{zz}}{{g_z}^2} + \frac{z^2(\mu-r)^2 g_{zz}}{2k^2 s^{2\beta}} -$

$(\mu - r)szg_{sz} = 0, \quad (28)$

so that

$${\pi_1}^* = -\frac{[(\mu-r)zg_z - k^2 s^{2\beta+1}g_s]}{gk^2 s^{2\beta}}. \quad (29)$$

5. Optimal investment strategy for specific utility

Assume the contributor takes an exponential utility

$$U(x) = -\frac{1}{q}e^{-qx}, \quad q > 0. \quad (30)$$

The absolute risk aversion of a decision maker with the utility described in (30) is a constant CARA utility. Since $g(T,s,z)=(U')^{-1}(z)$ and the CARA utility function we obtain

$$g(T,s,z)=-\frac{1}{q}lnz. \quad (31)$$

Hence we conjecture a solution to (28) of the following form

$$g(t,s,z)=-\frac{1}{q}[y(t)(lnz+m(t,s))]+w(t) \quad (32)$$

With boundary conditions

$$y(T)=1,\ w(T)=0, m(T,s)=0,$$

$$g_t=-\frac{1}{q}[y'(t)(lnz+m(t,s))+um_t]+w'(t),$$

$$g_s=-\frac{1}{q}ym_s,\ g_z=-\frac{y}{qz}\ ,\ g_{zz}=\frac{y}{qz^2}, g_{ss}=$$

$$-\frac{1}{q}ym_{ss},\ g_{sz}=0. \quad (33)$$

Substituting (33) into (28) gives;

$$[y'(t)-ry(t)]lnz+[-w'(t)+rw(t)+\theta a]q$$

$$+[m_t+rsm_s+\frac{1}{2}k^2s^{2\beta+2}m_{ss}+\frac{(\mu-r)^2}{2k^2s^{2\beta}}-rm+$$

$$\frac{y'}{y}m-r-\frac{1}{2}b^2]y=0,$$

such that

$$y'(t)-ry(t)=0, \quad (34)$$

$$m_t+rsm_s+\frac{1}{2}k^2s^{2\beta+2}m_{ss}+\frac{(\mu-r)^2}{2k^2s^{2\beta}}-$$

$$r-\frac{1}{2}b^2=0, \quad (35)$$

and

$$-w'(t)+rw(t)+\theta a=0. \quad (36)$$

Solving (34) and (36), we have

$$y(t)=e^{-r(t-T)} \quad (37)$$

and

$$w(t)=-\frac{\theta a}{r}(1-e^{-r(t-T)}). \quad (38)$$

Next we conjecture a solution of (35) with the following structure

And

$$m(t,s)=F(t)+H(t)s^{-2\beta},$$

$$F(T)=0, H(T)=0 \quad (39)$$

$$m_t=F_t+H_ts^{-2\beta}, m_s=-2\beta Hs^{-2\beta-1}, m_{ss}=$$

$$2\beta(2\beta+1)Hs^{-2\beta-2} \quad (40)$$

Substituting (40) into (35) we have

$$F_t+\beta(2\beta+1)k^2H-r-\frac{1}{2}b^2+s^{-2\beta}\Big[H_t-$$

2rβH+μ−r22k2=0, (41)

Such that

$$F_t+\beta(2\beta+1)k^2H-r-\frac{1}{2}b^2=0, \quad (42)$$

and

$$H_t-2r\beta H+\frac{(\mu-r)^2}{2k^2}=0. \quad (43)$$

Solving (43) with the given condition we get

$$H(t)=\frac{(\mu-r)^2}{4k^2r\beta}[1-e^{2r\beta(t-T)}]. \quad (44)$$

Next substituting (44) into (42) and solving (42) with the given condition we have

$$F(t)=\left[\frac{(2\beta+1)(\mu-r)^2}{4r}-r-\frac{1}{2}b^2\right](T-t)-$$

$$\left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1-e^{2r\beta(t-T)}\right)\right]. \quad (45)$$

Hence the solution to (28) for CARA utility function is given as

$$g_1(s,t,z)=-\frac{1}{q}\left[y(t)\left(lnz+m_1(t,s)\right)\right]+w(t),$$

Where

$$y(t)=e^{r(t-T)},$$

$$w(t)=-\frac{\theta a\,[1-e^{-r(T-t)}]}{r},$$

$$m_1(t,s)=\left[\frac{(2\beta+1)(\mu-r)^2}{4r}-r-\frac{1}{2}b^2\right](T-t)$$

$-\left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1-e^{2r\beta(t-T)}\right)\right]+\left[\frac{s^{-2\beta}(\mu-r)^2}{4k^2r\beta}\right.(1-$

e2rβt−T. (46)

The optimal investment strategy is given as

$$\pi_1{}^* = -\frac{[(\mu-r)zg_z - k^2s^{2\beta+1}g_s]}{gk^2s^{2\beta}}$$

Where

$$g_z = -\frac{1}{qz}e^{r(t-T)}, \quad (47)$$

$$g_s = \frac{1}{q}e^{r(t-T)}\frac{s^{-2\beta-1}(\mu-r)^2}{2k^2r}\left(1-e^{2r\beta(t-T)}\right)], \quad (48)$$

and

$$\pi_1{}^* = \frac{1}{q}\frac{(\mu-r)}{k^2s^{2\beta}g}e^{r(t-T)}$$

$$[1+\frac{(\mu-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)]. \quad (49)$$

if $ks^{\beta} = \sigma_*$, $M(\sigma_*) = \frac{(\mu-r)}{q\sigma_*{}^2}$, $N(t) = 1 + \frac{(\mu-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)$, $g = v$, Then

$$\pi_1{}^* = g_1{}^{-1}\,M(\sigma_*)\,N(t) \quad (50)$$

Similarly, The Jacobi Hamilton-Jacobi-Bellman (HJB) equation associated with (10) is

$H_t + \mu sH_s + (rv+\theta a)H_v + \frac{1}{2}k^2s^{2\beta+2}\left[H_{ss} -\right.$

Hvs2Hvv−12b2Hvv

$$-\frac{z^2(\mu-r)^2}{2k^2s^{2\beta}}\frac{H_v{}^2}{H_{vv}} - (\mu-r)s\frac{H_vH_{vs}}{H_{vv}} = 0 \quad (51)$$

With

$$\pi_2{}^* = -\frac{[(\mu-r)H_v + k^2s^{2\beta+1}H_{vs}]}{k^2s^{2\beta}vH_{vv}}. \quad (52)$$

Applying legendary transform to equation (51) and (52) we have;

$$g_t + rsg_s - (rg+\theta a) + \frac{1}{2}k^2s^{2\beta+2}g_{ss} + \left(\frac{(\mu-r)^2}{k^2s^{2\beta}} - r\right)zg_z - \frac{1}{2}b^2\frac{g_{zz}}{g_z{}^2} + \frac{z^2(\mu-r)^2g_{zz}}{2k^2s^{2\beta}} - (\mu-r)szg_{sz} = 0. \quad (53)$$

$$\pi_2{}^* = -\frac{[(\mu-r)zg_z - k^2s^{2\beta+1}g_s]}{gk^2s^{2\beta}} \quad (54)$$

Solving (53) for the CARA utility function given in (30) we obtain the following result

$$g_2(s,t,z) = -\frac{1}{q}\left[y(t)\left(lnz + m_2(t,s)\right)\right] + w(t)$$

with

$$y(t) = e^{r(t-T)},\ w(t) = -\frac{\theta a\left[1-e^{-r(T-t)}\right]}{r},$$

and

$$m_2(t,s) = \left[\frac{(2\beta+1)(\mu-r)^2}{4r} - r + \frac{1}{2}b^2\right](T-t)$$

$-\left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1-e^{2r\beta(t-T)}\right)\right]+\left[\frac{s^{-2\beta}(\mu-r)^2}{4k^2r\beta}\right.(1-$

e2rβt−T. (55)

The optimal investment strategy is given as

$$\pi_2{}^* = g_2{}^{-1}\,M(\sigma_*)\,N(t). \quad (56)$$

Proposition 5.1

Suppose

$b>0, V_1 > V_2$ *and* $m_1 < m_2$, *then* $V_1\pi_1{}^* > V_2\pi_2{}^*$

Proof

Since $b>0$, $m_1 < m_2$, then $g_1 < g_2$ and $\frac{1}{g_1} > \frac{1}{g_2}$.

At such,

$\frac{1}{g_1}M(\sigma_*)\,N(t) > \frac{1}{g_2}M(\sigma_*)\,N(t)$.

Since

$\frac{1}{g_1}M(\sigma_*)\,N(t) = \pi_1{}^*$ and $\frac{1}{g_2}M(\sigma_*)\,N(t) = \pi_2{}^*$,

it then implies that

$\pi_1{}^* > \pi_2{}^*$.

Also

$V_1 > V_2$.

Therefore

$V_1\pi_1{}^* > V_2\pi_2{}^*$.

Remark

1. From proposition (5.1), the surplus for investment with the death retires wealth is greater compared to when the death retires wealth is not part of the surplus for investment.
2. The proportion of the wealth to be invested in the risky asset is greater in the case when the death pensioners wealth is part of the surplus for investment
3. The fund manager stand a chance of making more interest with the surplus from the death pensioners wealth and stand a risk of losing more if the investment fails

6. Conclusion

The effect of death retirees' wealth in determining the optimal investment strategies for DC pension fund with multiple contributors was studied by modifying the model developed in Dawei and Jingyi (2014) and solved the optimized problem when the wealth of the death pensioners are added to the surplus to be invested and when their wealth are not added to the surplus to be investment via Legendre transformation and dual theory to obtain the explicit solution for CARA utility function.

We observed that the proportion of the wealth to be invested in the risky asset is greater in the case when the death pensioners wealth is part of the surplus for investment and also the fund manager stand a chance of making more interest with the surplus from the death retirees' wealth and stand a risk of losing more if the investment fails.

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