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Risky Asset Prices and Reinsurer's Surplus with Linear and Quadratic Expected Rate of Returns under Environmental Noise

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Abstract

This research paper presents a reinsurer's (RI) portfolio which is a combination of the RI surplus, a risk-free asset and a risky asset. We considered linear and quadratic cases for the expected rate of returns from the risky asset with environmental noise. Ito's lemma and maximum principle were used to determine the closed-form solutions of the risky assets for all considered cases. Also, the closed-form solutions of the RI's surplus were obtained for both cases. Furthermore, the relationship between the surplus process, time and the random environmental noise was also given. Finally, some numerical simulations were presented to study the impact of the expected rate of return and instantaneous volatilities of the stock market prices on the risky asset and also the behaviour of the RI's surpluses with time. It was observed that the price process of the risky asset is directly proportional to the expected rate of return and environmental noise, inversely proportional to the instantaneous volatilities and the risk-free interest rate while the surplus process is not directly dependent on the expected returns, instantaneous, volatility, risk-free interest rate, random noise but mostly dependent on the RI's safety loading and the number of claims to be serviced at any given study. Furthermore, we recommend that both insurer and the reinsurer should do diligent investigations before embarking on any investment in risky asset; especially asset with high appreciation rate to avoid being trap and are unable to pay claims to their client in the case of eventuality. Also, there should be a balance between the surplus, investment in risk-free asset and the risky asset to avoid being ruined.

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Introduction

The RI's strategy is a vital tool used by most insurance companies to manage risks involved in managing the companies away from ruin and payment of claims to their clients. According to (Chunxiang *et al*, 2018; Xiao *et al*, 2019; Malik *et al*, 2020; Njoku and Akpanibah, 2024), the RI is always involved in double risks which involve risk from investment and risk involved in the payment of claims. Hence, to properly manage the risk involved in the payment of claims to clients when the need arises, the insurer is permitted to buy reinsurance contracts from the reinsurer and move a specific percentage of his risk of the claim payment to the RI, since the RIs are seen as risk-lovers (Xiao *et al*, 2019). In general, Insurance can be viewed as a risk transfer tool but can also be utilized to mitigate risk. Due to the significance and the growing interest in portfolio optimization and reinsurance issues, numerous researchers have conducted extensive studies in this area. (Gu *et al*, 2012; Ihedioha, 2015; Akpanibah Osu, 2017).

Considering the importance of the RI, there is a need to study the surplus process of these reinsurance companies, the risky asset prices of the RI and the investment plan of the RI. The RI's strategy for maximizing the product of the insurer and the RI's utilities under a CEV model was studied by (Li *et al*, 2014); they obtained this by optimizing the exponential utility functions in which the claim process followed the GBM model. In (Ihedioha, 2015; Sheng, 2016), the optimal reinsurance and investment problem of maximizing the expected power utility function was investigated using a promotional budget. Some other authors such as (Egbe *et al*, 2013; Deng, 2019; Amadi, 2022; Njoku and Akpanibah, 2024), studied the RI's strategies and the surplus process of the RI under different assumptions. Also, the optimal RI's strategy was studied for an insurer whose premium was stochastic nature by (Akpanibah and Osu, 2017); they used the Legendre transformation approach to find the optimal RI's strategy under exponential utility. The Jump diffusion process was used to model their RI's problem and

obtained the RI's strategy under exponential utility by (Lin and Li, 2011; Wang *et al*, 2018).
(Osu and Ihedioha, 2013; Sultan and Aqsa, 2017), solved the RI's problem and obtained the RI's strategy; furthermore, the probability of ruin of the RI was determined. (Ini et al, 2021), investigated a mathematical model for an insurer's and RI's portfolio and used the model to determine the investment strategies when the risky asset was modelled using the constant elasticity of variance model while (Malik *et al*, 2020), used the fractional power utility function to determine the RI's investment strategy under the CEV model; it was observed that higher value of the RI's safety loading may leadto a decrease in the RI's strategy. Hence to maintain a stable income, the insurer should buy a less reinsurance policy. (Chunxiang *et al*, 2018) studied optimal excess of loss reinsurance and investment problems with delay and jump-diffusion risk.
However, the need to examine the stochastic nature of the price changes of the risky asset which plays a crucial role in determining the RI's strategy and the surplus process has become extremely necessary. Hence, some researchers such as (Davies et al, 2019), studied the stability analysis of price changes on the floor of a stock market and outlined necessary steps in the derivation and determination of equilibrium prices and growth rate of stock market prices. (Adeosun, 2015; Akpanibah and Ogunmodimu, 2023), studied the unstable nature of stock market prices and stochastic analysis of the behaviour of stock prices. Results show that the model is efficient for the prediction of stock prices.
(Osu *et al*, 2022), showed that the risky asset prices, risk-free asset prices and over-plus of company's claim which changes over time between the one who insures and reinsures. Based on these changes, some sources of randomness were introduced and solve for time-varying investment returns. Hence, in this paper, we solve for the RI's surplus under exponential utility and the price process of the risky assets to determine the behavior of the assets before investment.

Materials and Methods

Itô's Process

Definition 2.1.1 $\mathcal{Q}_{\mathcal{G}}(S,T)$denotes the class of processes $g(t,\omega) \in \Re$ satisfying:
1. $(t,\omega) \to g(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$;
2. There exists an increasing family of σ-algebras $\mathcal{G}(t)$with $t \geq 0$, such that $\mathcal{B}_0$is a martingale with respect to $\mathcal{G}(t)$and that $g(t)$ is $\mathcal{G}(t)$-adapted;
3.$\mathbb{p}\left[\int_S^T g(s,\omega)^2\, ds < \infty\right] = 1.$

Definition 2.1.2 Let $\mathcal{B}_t$Be a one-dimensional Brownian motion on $(\Omega, F, \mathbb{p})$. A (one-dimensional) Itô process (or stochastic integral) is a stochastic process $X(t)$on $(\Omega, F, \mathbb{p})$ of the form

$$\mathcal{D}(t) = \mathcal{D}(0) + \int_S^T a(s,\omega)\, ds + \int_S^T \sigma(s,\omega)\, d\mathcal{B}_t \qquad (1)$$

Where $\sigma \in \mathcal{Q}_{\mathcal{G}}$so that

$$\mathbb{p}\left[\int_S^T a(s,\omega)^2\, ds < \infty, \quad \forall \ \ t \geq 0\right] = 1.$$

We also assume that a is $\mathcal{G}(t)$ −adapted, where $\mathcal{G}(t)$is an increasing family of σ-algebras,
$\{\mathcal{G}(t)\}t \geq 0$, such that $\mathcal{B}_t$is a martingale with respect to $\mathcal{G}(t)$, and

$$\left[\int_S^T a(s,\omega)^2\, ds < \infty, \quad \forall \ \ t \geq 0\right] = 1.$$

Let $\mathcal{D}(t)$ be an Itô process in the of form (1), the differential form of (1) is given as

$$d\mathcal{D}(t) = a(s,\omega)dt + \sigma(s,\omega)d\mathcal{B}_t \qquad (2)$$

Where a is the drift and σ, the standard deviation representing the instantaneous volatility.

Remark 1. For Ito processes $U(t)$and $V(t)$in $\Re$, Itô's product rule gives
$d(\mathcal{X}(t)\mathcal{Y}(t)) = \mathcal{X}(t)d\mathcal{Y}(t) + \mathcal{Y}(t)d\mathcal{X}(t) + d\mathcal{X}(t)d\mathcal{Y}(t).$

Remark 2. Let $\mathcal{D}(t)$ be an Itô process given by (2)
Let $g(t,\mathcal{D}(t)) \in C^2([0,\infty) \times \Re)$.Then $g(t,\mathcal{D}(t))$ is again an Itô process, and

$$dg(t,\mathcal{D}(t)) = \begin{bmatrix} \frac{\partial g}{\partial t}\big(t,\mathcal{D}(t)\big)dt + \frac{\partial g}{\partial x}\big(t,\mathcal{D}(t)\big)d\mathcal{D}(t) \\ +\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,\mathcal{D}(t))(d\mathcal{D}(t))^2 \end{bmatrix}, (3)$$

where$(d\mathcal{D}(t))^2 = d\mathcal{D}(t)d\mathcal{D}(t)$is computed according to the rules $d\mathcal{B}_t d\mathcal{B}_t = dt$,
$dtdt = dtd\mathcal{B}_t = d\mathcal{B}_t dt = 0.$

Model Formulations

Reinsurer's Surplus Process

From *(*Cao and Wan 2009; Li et al, 2015), the claim process $\mathcal{V}(t)$ which follows the GBM is given as follows

$$d\mathcal{V}(t) = v_1 dt - v_2 d\mathcal{B}_0(t), \qquad (4)$$

where $v_1 and\ v_1$are positive constants, and $\mathcal{B}_0(t)$is a standard Brownian motion defined on the complete probability space (Ω; $\mathcal{F}_t$; $\mathcal{P}$*).* From the expected value principle in *(*Nozadi, 2014), an insurer's premium rate is given as

$$v = (1+\vartheta)v_1 \qquad (5)$$

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where $\theta > 0$is the safety loading of an insurer. From (Nozadi, 2014), a classical Cramer-Lundberg model for the surplus process is given as follows

$$\mathrm{d}\mathcal{N}(\mathrm{t}) = c_0 + \upsilon t - \mathcal{V}(t) \quad \mathrm{t} \geq 0, \tag{6}$$

where $\mathcal{N}(\mathrm{t})$and $c_0$are the insurers capital at time $t$and initial capital $\mathcal{N}(0) = c_0$, respectively. According to (4), (5) and (6), the surplus process for the insurer is given as

$$\frac{\mathrm{d}\mathcal{N}(\mathrm{t})}{\mathcal{N}(\mathrm{t})} = \upsilon dt - d\mathcal{V}(t) = \upsilon_1 \vartheta dt + \upsilon_2 d\mathcal{B}_0(t). \tag{7}$$

Furthermore, the insurer can buy reinsurance contract to reduce risk. Suppose the insurer pays reinsurance premium continuously at

$$\hbar_1 = (1+\eta)\upsilon_1 \tag{8}$$

where $\eta > \vartheta > 0$is the safety loading of the reinsurer. Hence, $\mathcal{N}_1(t)$ representing the reinsurer's surplus is given by

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \upsilon dt - \big(1 - n(t)\big)d\mathcal{V}(t) - \hbar_1 n(t)dt \tag{9}$$

Substituting (4), (5) and (8) into (9), we have

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \big(\vartheta - \eta n(t)\big)\upsilon_1 dt + \upsilon_2\big(1 - n(t)\big)d\mathcal{B}_0(t). \tag{10}$$

where $n(t)$ is proportion reinsurance at time t.

Financial Market

Assume $\{\mathcal{B}_1(t): t \geq 0\}$ is a standard Brownian motion which is defined on a complete probability space (Ω, F, P) such that Ω is a real space, P a probability measure and F a filtration representing information generated by the Brownian motion. Also, consider a reinsurer with a portfolio consisting of one risk-free asset and one risky asset in a continuously opened financial market over the interval $t \in [0, T]$, where T is the terminal period for investment.
Suppose $\mathcal{H}_0(t)$ is the price process of the risk-free asset at time t, then the financial model is given as follows

$$\begin{cases} \frac{d\mathcal{H}_0(t)}{\mathcal{H}_0(t)} = r(t)dt \\ \mathcal{H}_0(0) = \hbar_0 > 0 \end{cases}. \tag{11}$$

where r is the predetermined interest rate process
Let $\mathcal{H}_1(t)$ denote the price of the stocks described by the GBM model is given by the stochastic differential equations at $t \geq 0$ as follows

$$\begin{cases} \frac{d\mathcal{H}_1(t)}{\mathcal{H}_1(t)} = \mu\rho dt + \sigma(t)d\mathcal{B}_1(t) \\ \mathcal{H}_1(0) = \hbar_0 \end{cases}. \tag{12}$$

Also, Let $\mathcal{H}_2(t)$ denote the price of the risky asset described by the GBM model when the expected rate of return is quadratic is given as follows

$$\begin{cases} \frac{d\mathcal{H}_2(t)}{\mathcal{H}_2(t)} = \rho\mu^2 dt + \sigma(t)d\mathcal{B}_1(t) \\ \mathcal{H}_1(0) = \hbar_0 \end{cases} \tag{13}$$

Where μ is the appreciation rate of the stock, σ is the instantaneous volatility of the risky asset and ρ is random noise due to environmental effects.

Optimization Problem

Suppose $\ell(t)$ represents the reinsurer's optimal investment plan such that the utility attained by the reinsurer from a given state c at time t is modeled thus

$$\mathcal{N}_\ell(t,c) = E_\ell\big[U\big(\mathcal{C}(T)\big) \mid \mathcal{C}(t) = c\big], \tag{14}$$

where t is the time, r is the risk-free interest rate and c is the wealth. The objective here is to determine the optimal investment plan and the optimal value function of the investor given as ℓ^* and $\mathcal{N}(t,c) = \sup_\ell \mathcal{N}_\ell(t,c)$ respectively such that

$$\mathcal{N}_{\ell^*}(t,c) = \mathcal{N}(t,c). \tag{15}$$

Let $\mathcal{C}(t)$ be the insurer's wealth at time t and the amount of wealth of insurer invested on risky asset at time t denoted by $\ell(t)$ such that the remainder $\mathcal{C}(t) - \ell(t)$ is invested in risk-free asset. Then the differential form associated with the fund size of the reinsurer is given as:

$$d\mathcal{C}(t) = (\mathcal{C}(t) - \ell)\frac{d\mathcal{H}_0(t)}{\mathcal{H}_0(t)} + \ell\frac{d\mathcal{H}_1(t)}{\mathcal{H}_1(t)} + \frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} \tag{16}$$

substituting (10), (11), (12) and (13) into (16), we have

$$d\mathcal{C}(t) = \begin{pmatrix} \begin{pmatrix} r\mathcal{C}(t) + \ell(\mu\rho - r) \\ +\big(\vartheta - \eta n(t)\big)\upsilon_1 \end{pmatrix} dt \\ +\upsilon_2\big(1 - n(t)\big)d\mathcal{B}_0(t) + \ell\sigma(t)d\mathcal{B}_1(t) \end{pmatrix}, \tag{17}$$

$$d\mathcal{C}(t) = \begin{pmatrix} \begin{pmatrix} r\mathcal{C}(t) + \ell(\mu^2\rho - r) \\ +\big(\vartheta - \eta n(t)\big)\upsilon_1 \end{pmatrix} dt \\ +\upsilon_2\big(1 - n(t)\big)d\mathcal{B}_0(t) + \ell\sigma(t)d\mathcal{B}_1(t) \end{pmatrix}, \tag{18}$$

Equation (17) and (18) represents the reinsurer's wealth for linear and quadratic appreciation rate of the reinsurer.

Applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (17) is obtained by maximizing (14) subject to (17) to obtain the HJB equation for linear appreciation rate.

$$\left\{\begin{array}{c}\mathcal{N}_t + \left[rc + \ell(\mu\rho - r) + \left(\vartheta - \eta n(t)\right)v_1\right]\mathcal{N}_c \\ + \frac{1}{2}\left({v_2}^2\left(1 - n(t)\right)^2 + \ell^2\sigma^2\right)\mathcal{N}_{cc}\end{array}\right\} = 0. \quad (19)$$

To obtain the first order maximizing condition for equation (19), we differentiate (19) with respect to the two control variables $\ell(t)$ and $n(t)$ as follows

$$\ell^* = -\frac{(\mu\rho - r)\mathcal{N}_c}{\sigma^2\mathcal{N}_{cc}} \quad (20)$$

$$n^* = \frac{\eta v_1}{{v_2}^2}\frac{\mathcal{N}_c}{\mathcal{N}_{cc}} + 1 \quad (21)$$

Substituting (20) and (21) into (19), we have

$$\mathcal{N}_t + [rc + (\vartheta - \eta)v_1]\mathcal{N}_c - \frac{1}{2}\left[\begin{array}{c}\frac{(\mu\rho - r)^2}{\sigma^2} \\ -\frac{\eta^2{v_1}^2}{{v_1}^2}\end{array}\right]\frac{\mathcal{N}_c^2}{\mathcal{N}_{cc}} = 0 \quad (22)$$

Since we are interested in obtaining the over plus of the reinsurer, we will need to find the solution for n^* by solving (22) but from (Njoku and Akpanibah, 2024),

$$n^* = \left(1 - \frac{\eta v_1}{{v_2}^2}\right)\left(\frac{1}{z^2}\right) = \left(1 - \frac{\eta v_1}{{v_2}^2}\right)\left(\frac{1}{z}\right)^2$$

Substituting we have

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)} = \left\{\begin{array}{c}\left(\frac{(\theta z^2 - \eta)v_1{v_2}^2 + \eta^2 a^2}{z^2{v_2}^2}\right)dt \\ + \left(\frac{(z^2 - 1){v_2}^2 + \eta v_1}{z^2 b}\right)d\mathcal{B}_0(t) \\ \mathcal{N}_1(0) = 1\end{array}\right. \quad (23)$$

Results

In this section, we are interested in solving the prices of risk-free assets and risky assets under different assumptions. We will give solutions to the stochastic differential equations in (12), (13) and (23)

Solution of Price Process of Risky Asset for Stochastic Case with Linear Rate of Expected Returns

Recall that the price process of the risky asset with linear rate of expected rate of return is given by (12) as

$$\left\{\begin{array}{l}\frac{d\mathcal{H}_1(t)}{\mathcal{H}_1(t)} = \rho\mu dt + \sigma(t)d\mathcal{B}_1(t) \\ \qquad \mathcal{H}_1(0) = e^{-t}\end{array}\right.$$

To solve (12), we apply the Itô process in (3) as follows

Let

$$g\left(t, \mathcal{H}_1(t)\right) = ln\mathcal{H}_1(t)\,, \quad (24)$$

Then from Itô process,

$$dg\left(t, \mathcal{H}_1(t)\right) = \left[\begin{array}{c}\frac{\partial g\left(t,\mathcal{H}_1(t)\right)}{\partial t}dt + \frac{\partial g\left(t,\mathcal{H}_1(t)\right)}{\partial \mathcal{H}_1(t)}d\mathcal{H}_1(t) \\ + \frac{1}{2}\frac{\partial^2 g\left(t,\mathcal{H}_1(t)\right)}{\partial \mathcal{H}_1(t)^2}\left(d\mathcal{H}_1(t)\right)^2\end{array}\right], \quad (25)$$

where $(d\mathcal{H}_1(t))^2 = d\mathcal{H}_1(t)d\mathcal{H}_1(t)$ is computed according to the rules

$$dtdt = dtd\mathcal{B}_1 = d\mathcal{B}_1 dt = 0;\; d\mathcal{B}_1 d\mathcal{B}_1 = dt.$$

From (12),

$$d\mathcal{H}_1(t) = \mathcal{H}_1(t)\left(\rho\mu dt + \sigma(t)d\mathcal{B}_1(t)\right) \quad (26)$$

Substituting (26) into (25), we have

$$dg\left(t, \mathcal{H}_1(t)\right) = \left(\begin{array}{c} g_t dt \\ + g_{\mathcal{H}_1}\mathcal{H}_1(t)\left(\begin{array}{c}\rho\mu dt \\ + \sigma(t)d\mathcal{B}_1(t)\end{array}\right) \\ + \frac{1}{2}g_{\mathcal{H}_1\mathcal{H}_1}\sigma^2\mathcal{H}_1^2 dt\end{array}\right), \quad (27)$$

Differentiating (24), we have

$$\left\{\begin{array}{l} g_t = 0 \\ g_{\mathcal{H}_1} = \frac{1}{\mathcal{H}_1} \\ g_{\mathcal{H}_1\mathcal{H}_1} = -\frac{1}{\mathcal{H}_1^2}\end{array}\right. \quad (28)$$

Substituting (28) into (27), we have

$$dg\left(t, \mathcal{H}_1(t)\right) = \left(\begin{array}{c}0dt + \frac{1}{\mathcal{H}_1}\mathcal{H}_1(t)\left(\rho\mu dt + \sigma(t)d\mathcal{B}_1(t)\right) \\ - \frac{1}{2\mathcal{H}_1^2}\sigma^2\mathcal{H}_1^2 dt\end{array}\right), \quad (29)$$

$$dg\left(t, \mathcal{H}_1(t)\right) = (\rho\mu - \tfrac{1}{2}\sigma^2)dt + \sigma(t)d\mathcal{B}_1, \quad (30)$$

Integrating both sides, we have

$$g\left(t, \mathcal{H}_1(t)\right) = (\rho\mu - \tfrac{1}{2}\sigma^2)t + \sigma(t)\mathcal{B}_1 \quad (31)$$

From (27),

$$\mathcal{H}_1(t) = \mathcal{H}_1(0)Exp\left[(\rho\mu - \tfrac{1}{2}\sigma^2)t + \sigma(t)\mathcal{B}_1\right] \quad (32)$$

But $\mathcal{H}_1(0) = e^{-t}$, hence

$$\mathcal{H}_1(t) = Exp\left[(\rho\mu - \tfrac{1}{2}\sigma^2 - 1)t + \sigma(t)\mathcal{B}_1\right] \quad (33)$$

Solution of Price Process of Risky Asset for Stochastic Case with Quadratic Rate of Expected Returns

Recall that the price process of the risky asset with linear rate of expected rate of return is given by (13) as

$$\left\{\begin{array}{l}\frac{d\mathcal{H}_2(t)}{\mathcal{H}_2(t)} = \rho\mu^2 dt + \sigma(t)d\mathcal{B}_1(t) \\ \qquad \mathcal{H}_2(0) = e^{-t}\end{array}\right.$$

To solve (13), we apply the Itô process in (3) as follows

$$g\left(t, \mathcal{H}_2(t)\right) = ln\mathcal{H}_2(t)\,, \quad (34)$$

Then from Itô process,

$$dg\big(t,\mathcal{H}_2(t)\big)=\begin{bmatrix}\frac{\partial g(t,\mathcal{H}_2(t))}{\partial t}dt+\frac{\partial g(t,\mathcal{H}_2(t))}{\partial \mathcal{H}_2(t)}d\mathcal{H}_2(t)\\ +\frac{1}{2}\frac{\partial^2 g(t,\mathcal{H}_2(t))}{\partial \mathcal{H}_2(t)^2}(d\mathcal{H}_2(t))^2\end{bmatrix}, \quad (35)$$

where$(d\mathcal{H}_2(t))^2 = d\mathcal{H}_2(t)d\mathcal{H}_2(t)$is computed according to the rules

$$dtdt = dtd\mathcal{B}_1 = d\mathcal{B}_1dt = 0;\ d\mathcal{B}_1d\mathcal{B}_1 = dt.$$

From (13),

$$d\mathcal{H}_2(t) = \mathcal{H}_2(t)\big(\rho\mu^2dt+\sigma(t)d\mathcal{B}_1(t)\big) \quad (36)$$

Substituting (36) into (35), we have

$$dg\big(t,\mathcal{H}_2(t)\big)=\begin{pmatrix} g_t dt\\ +g_{\mathcal{H}_1}\mathcal{H}_2(t)\begin{pmatrix}\rho\mu^2dt\\ +\sigma(t)d\mathcal{B}_1(t)\end{pmatrix}\\ +\frac{1}{2}g_{\mathcal{H}_2\mathcal{H}_2}\sigma^2\mathcal{H}_2^2dt\end{pmatrix}, \quad (37)$$

Differentiating (37), we have

$$\begin{cases} g_t = 0\\ g_{\mathcal{H}_2} = \frac{1}{\mathcal{H}_2}\\ g_{\mathcal{H}_2\mathcal{H}_2} = -\frac{1}{\mathcal{H}_2^2}\end{cases} \quad (38)$$

Substituting (38) into (37), we have

$$dg\big(t,\mathcal{H}_2(t)\big) = (\rho\mu^2-\tfrac{1}{2}\sigma^2)dt+\sigma(t)d\mathcal{B}_1, \quad (39)$$

Integrating both sides, we have

$$g\big(t,\mathcal{H}_2(t)\big) = (\rho\mu^2-\tfrac{1}{2}\sigma^2)t+\sigma(t)\mathcal{B}_1 \quad (40)$$

From (34),

$$\mathcal{H}_2(t) = \mathcal{H}_2(0)Exp\left[(\rho\mu-\tfrac{1}{2}\sigma^2)t+\sigma(t)\mathcal{B}_1\right] \quad (41)$$

But $\mathcal{H}_2(0) = e^{-t}$, hence

$$\mathcal{H}_2(t) = Exp\left[(\rho\mu-\tfrac{1}{2}\sigma^2-1)t+\sigma(t)\mathcal{B}_1\right] \quad (42)$$

3.3 Solution of Surplus Process of the Reinsurer

Recall that the price process of the risky asset with linear rate of expected rate of return is given by (23) as

$$\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)}=\begin{cases}\left(\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\right)dt+\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathrm{d}\mathcal{B}_0(t)\\ \mathcal{N}_1(0)=1\end{cases}$$

Let $\big(t,\mathcal{N}_1(t)\big) = ln\mathcal{N}_1(t)$, (43)

Then from Itô process,

$$dg\big(t,\mathcal{N}_1(t)\big)=\begin{bmatrix}\frac{\partial g(t,\mathcal{N}_1(t))}{\partial t}dt+\frac{\partial g(t,\mathcal{N}_1(t))}{\partial \mathcal{N}_1(t)}d\mathcal{N}_1(t)\\ +\frac{1}{2}\frac{\partial^2 g(t,\mathcal{N}_1(t))}{\partial \mathcal{N}_1(t)^2}(d\mathcal{N}_1(t))^2\end{bmatrix}, \quad (44)$$

Where $(d\mathcal{N}_1(t))^2 = d\mathcal{N}_1(t)d\mathcal{N}_1(t)$ is computed according to the rules

$$dtdt = dtd\mathcal{B}_0 = d\mathcal{B}_0dt = 0;\ d\mathcal{B}_0d\mathcal{B}_0 = dt.$$

From (23),

$$d\mathcal{N}_1(t) = \mathcal{N}_1(t)\begin{pmatrix}\left(\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\right)dt\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathrm{d}\mathcal{B}_0(t)\end{pmatrix} \quad (45)$$

Substituting (45) into (44), we have

$$dg\big(t,\mathcal{N}_1(t)\big)=\begin{pmatrix} g_t dt\\ +g_{\mathcal{N}_1}\mathcal{N}_1(t)\begin{pmatrix}\left(\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\right)dt\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathrm{d}\mathcal{B}_0(t)\end{pmatrix}\\ +\frac{1}{2}g_{\mathcal{N}_1\mathcal{N}_1}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2{\mathcal{N}_1}^2dt\end{pmatrix}, \quad (46)$$

Differentiating (43), we have

$$\begin{cases} g_t = 0\\ g_{\mathcal{N}_1} = \frac{1}{\mathcal{N}_1}\\ g_{\mathcal{N}_1\mathcal{N}_1} = -\frac{1}{\mathcal{N}_1^2}\end{cases} \quad (47)$$

Substituting (47) into (46), we have

$$dg\big(t,\mathcal{N}_1(t)\big)=\begin{bmatrix}\begin{pmatrix}\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\\ -\frac{1}{2}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2\end{pmatrix}dt\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathrm{d}\mathcal{B}_0(t)\end{bmatrix}, \quad (48)$$

Integrating both sides, we have

$$g\big(t,\mathcal{N}_1(t)\big)=\begin{bmatrix}\begin{pmatrix}\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\\ -\frac{1}{2}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2\end{pmatrix}t\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathcal{B}_0\end{bmatrix} \quad (49)$$

From (43), we have

$$ln(\tfrac{\mathcal{N}_1(t)}{\mathcal{N}_1(0)})=\begin{pmatrix}\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\\ -\frac{1}{2}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2\end{pmatrix}t+\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathcal{B}_0$$

$$\mathcal{N}_1(t)=\mathcal{N}_1(0)Exp\begin{bmatrix}\begin{pmatrix}\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\\ -\frac{1}{2}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2\end{pmatrix}t\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathcal{B}_0\end{bmatrix}$$

But $\mathcal{N}_1(0) = 1$, hence

$$\mathcal{N}_1(t)=Exp\begin{bmatrix}\begin{pmatrix}\frac{(\theta z^2-\eta)v_1{v_2}^2+\eta^2{v_1}^2}{z^2{v_2}^2}\\ -\frac{1}{2}\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)^2\end{pmatrix}t\\ +\left(\frac{(z^2-1){v_2}^2+\eta v_1}{z^2v_2}\right)\mathcal{B}_0\end{bmatrix} \quad (50)$$

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Numerical Simulations

In this section, some numerical simulations are presented to study the impact of some sensitive parameters on the risky assets for linear and quadratic rates of returns and the reinsurer's surplus. To achieved this, the following data extracted from (He and Liang, 2013; Akpanibah and Ogheneoro, 2018; Akpanibah *et al*, 2020), will be used unless otherwise stated $r = 0.3$, $v_1 = 1.5$, $v_2 = 1, \sigma = 0.1$, $\eta = 2, \mu = 0.5$, $T = 10, \theta = 0.5, z = 0.1, \rho = 0.01, \mathcal{B}_0 = 1$

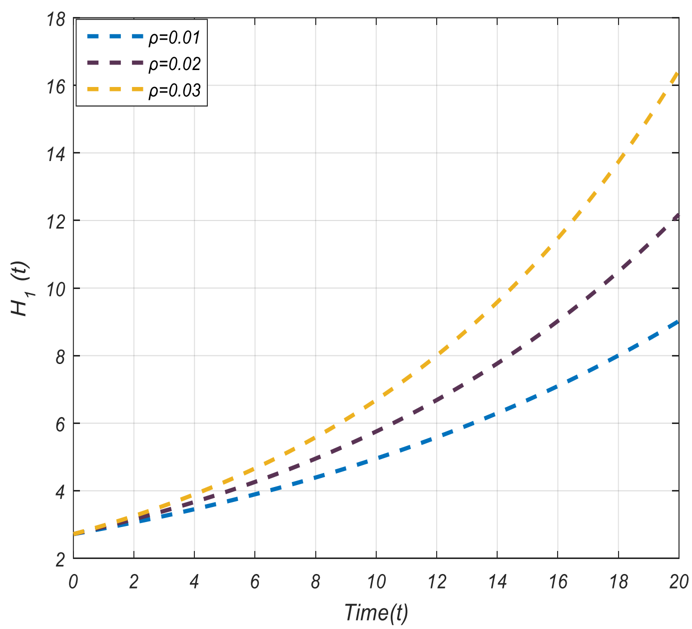

Figure 1: Time evolution of risky asset $\mathcal{H}_1(t)$ with linear appreciation rate and different random noise

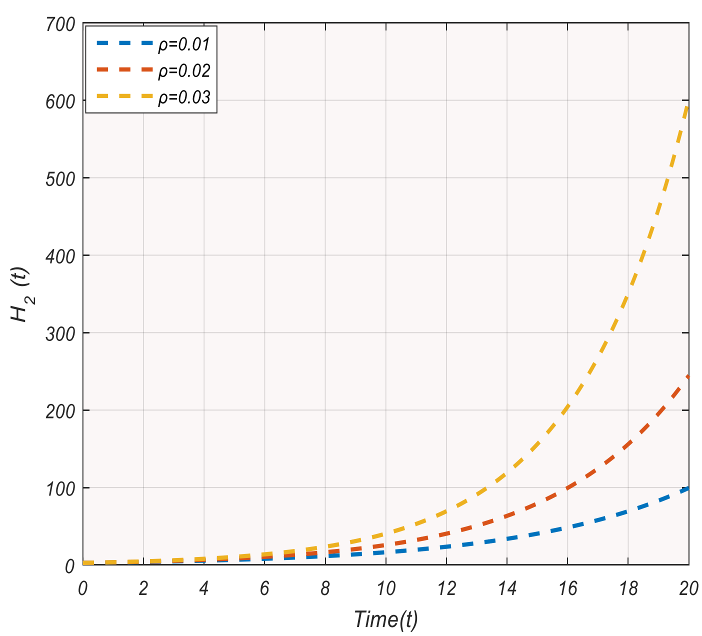

Figure 2: Time evolution of risky asset $\mathcal{H}_2(t)$ quadratic appreciation rate and different random noise

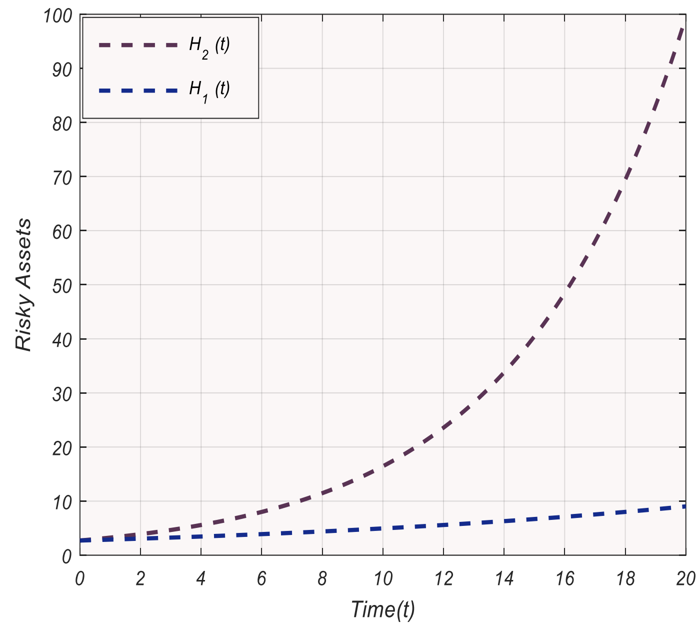

Figure 3: Time evolution of the risky assets with linear and quadratic appreciation rate

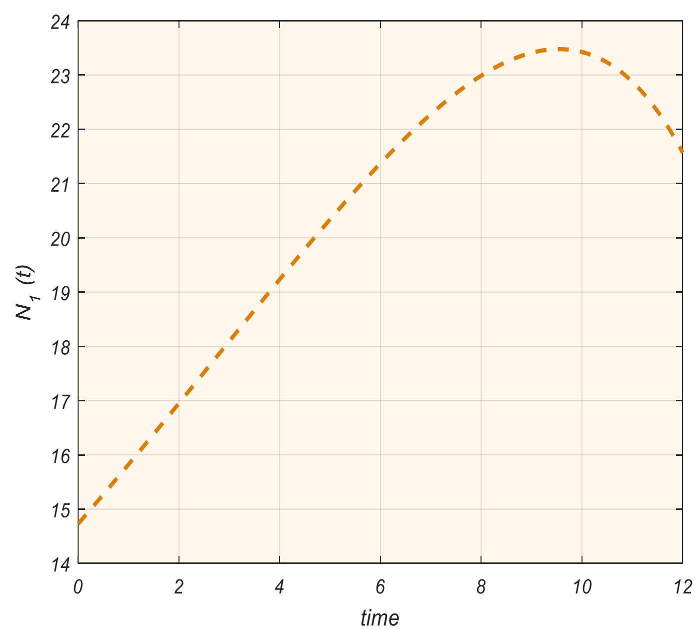

Figure 4: The evolution of $N_1(t)$with time

Discussion

Figure 1, shows the evolution of the price of the risky asset $\mathcal{H}_1(t)$of a reinsurer with a linear appreciation rate under different environmental noises. It was observed that the risky

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asset of the insurance company increases with time. Also, we observed that the risky asset increases with the environmental noise. This implies that the more the noise is in the financial market, it will affect the behaviour of such an asset. This is so because more noise such as war, inflation, pandemic and polical crisis make the asset very volatile and may affect the price of such asset. Hence, the insurance company should be mindful of the environmental noises when making investment decisions. In figure 2, the price process of the risky asset $\mathcal{H}_2(t)$of a reinsurer with quadratic appreciation rate under different environmental noises is presented. It was observed that the risky asset of the insurance company increases with time. Also, we observed similar to figure 1, that the risky asset increases with the environmental noise. This implies that the more noise there is in the financial market, it will affect the behaviour of such an asset. Hence, the insurer should be careful in investment choices in the presence of environmental noise. Figure 3, presents the time evolution of each of the risky asset prices with a linear appreciation rate of returns and quadratic rate of returns. We compared their prices and observed with a quadratic appreciation rate of returns, the price of such assets skyrocketed. This is so because most investors will embrace such an asset whose appreciation rate is very high and fast. Hence insurance companies should be very mindful of this type of asset because it also comes with higher risk. This is very consistent with (He and Liang, 2013; Akpanibah and Oghenero, 2014; Akpanibah *et al*, 2020), which used the efficient frontier to show the relationship between expectation and risk.

Figure 4, shows the evolution of the surplus process. $N_1(t)$ of a reinsurer. It was observed that the surplus process of the insurance company increases at the early stage of investment and as the need for payment of claims begins to set in, the surplus of the insurance company begins to go down. It is clear to see in the sense that funds are allocated towards payment of claims etc; which reduces the interest rate of the insurer and reinsurer at this time. We also observed that the surplus process is not directly dependent on the instantaneous volatility, risk-free interest, appreciation rate of the risky asset etc but depends on the safety loading and the wealth of the insurance company.

Conclusion

In conclusion, we considered the linear and quadratic expected rates of returns from the risky asset with environmental noise. The closed-form solutions for the risky assets and RI's surplus were obtained using Ito's lemma and maximum principle for both linear and quadratic rates of returns. Furthermore, the relationship between the surplus process, time and the random environmental noise were presented using numerical simulations. It was observed that the price process of the risky asset is directly proportional to the expected rate of return and environmental noise, inversely proportional to the instantaneous volatilities and the risk free interest rate while the surplus process does not necessarily depend on the expected returns, instantaneous, volatility, risk free interest rate, random noise but mostly dependent on the RI's safety loading and the amount of claims to be serviced at any given time. Finally, we recommend that both insurer and the reinsurer should do diligent investigations before embarking on any investment in risky asset; especially asset with high appreciation rate to avoid being trap and are unable to pay claims to their client in the case of eventuality. Secondly, there should be a balance between the surplus, investment in risk-free asset and the risky asset to avoid being ruined.

Conflict of Interests

The authors declare that there is no conflict of interest regarding this paper.

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