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Solving First Order Delay Differential Equations Using Block Simpson's Methods

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Abstract

This paper is concerned with solving first order delay differential equations (DDEs) using Block Simpson's Methods for step number k = 2, 3 and 4 without interpolation techniques in approximating the delay term. The discrete schemes were derived from the continuous formulations of the method through multistep collocation method by matrix inversion technique. The convergence and stability analysis of these discrete schemes were investigated. The implementation of these discrete schemes was carried out in block form to solve some first order delay differential equations. It was observed that the scheme for step number k = 4 performed better in terms of accuracy than the schemes for step number k = 3 and 2 respectively when compared with their exact solutions.

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Introduction

Considerable literatures exist for the normal k-step linear multi-step methods for the solution of delay differential equations (DDEs) of the form

$$\begin{cases} y'(t) = f\big(t, y(t), y(t-\tau)\big), & for\ t > t_0, \tau > 0 \\ \quad y(t) = \varphi(t), & for\ t \le t_0 \end{cases} \tag{1}$$

where $\varphi(t)$ is the initial function, τ is called the delay, $t-\tau$ is called the delay argument and $y(t-\tau)$ is the solution of the delay term. So many researchers such as Al-Mutib, (1977), Oberle, and Pesh(1981) used linear multistep methods to solve delay differential equations using interpolation techniques such as Hermite and Neville's interpolations in approximating the delay term. Ishak,et al (2008), Ishak et al. (2010) solved delay differential equations in predictor-corrector mode which were not self-starting and used interpolation techniques in approximating the delay term. Bocharov et al (1996) considered the application of the linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays and Nordsieck's interpolation technique was used to approximate the delay term. Majid et al (2013) solved delay differential equations by the five -point one-step block method using Neville's interpolation in approximating the delay term.

Sirisena and Yakubu (2019), used Reformulated Block Backward Differentiation Formulae Methods to solve delay differential equations which were reformulated by shifting the backward differentiation formulae methods one-step backward to produce discrete schemes of step numbers k = 3 and k = 4. They implemented these discrete schemes in block form to solve some delay differential equations without using any interpolation technique in approximating the delay term.

One of the major drawbacks of using these interpolation techniques by Majid et al (2013) is that, the order of these interpolating polynomials should be at least the same with that of the numerical method used which is very difficult to achieve, otherwise the accuracy of the method will not be preserved. It is expected that in the approximation of the delay term, using an accurate and efficient formula will be appropriate.

In order to circumvent this drawback of using interpolation techniques in which the order of the interpolating polynomials should be at least of the same as that of the numerical method used, Block Simpson's Methods for step number k = 2, 3 and 4 which are self-starting shall be presented as a simple form of linear multistep method and would be implemented to solve some first order DDEs. The block methods will be implemented using fixed step size and the delay term shall be approximated using the order of the sequence proposed by Sirsena *et al*.(2019) without using those known interpolation techniques such as Hermite, Nordsieck, Newton divided difference, Neville's interpolation etc. in order to preserve the desired accuracy.

Derivation Techniques

Derivation of Multistep Collocation Method

In Sirisena (1997) a *k*-step multistep collocation method with *m* collocation points was obtained as

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f(x, y(x)) \tag{2}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \text{ for } j = \{0,1,...,t-1\} \quad (3)$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \text{ for } j = \{0,1,...,m-1\} \quad (4)$$

where $x_0, \ldots, x_{m-1}$ are the m collocation points and x_{n+j}, j = 0, 1, 2, …, t – 1 are the t arbitrarily chosen interpolation points.

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena (1997) arrived at a matrix equation of the form

$$DC = I \quad (5)$$

where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \quad (6)$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \quad (7)$$

It follows from (5) that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (2).

Subsequently, the continuous formulations of the BSMs for step numbers k = 2, 3 and 4 shall be derived.

Derivation of Continuous Formulation of BSM for *K* = 2

Here, the number of interpolation points, t = 1 and the number of collocation points m = 3. Therefore, (2) become

$$y(x) = \alpha_0(x)y_n + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2}\right] \quad (8)$$

The matrix D in (5) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_n + 2h & 3(x_n+h)^2 \\ 0 & 1 & 2x_n + 4h & 3(x_n+2h)^2 \end{bmatrix} \quad (9)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the continuous scheme is obtained using (2) and evaluating it at $x = x_{n+1}$ and $x = x_{n+2}$, the following discrete schemes are obtained:

$$y_{n+1} = y_n + \frac{5}{12}hf_n + \frac{2}{3}hf_{n+1} - \frac{1}{12}hf_{n+2}$$

$$y_{n+2} = y_n + \frac{1}{3}hf_n + \frac{4}{3}hf_{n+1} + \frac{1}{3}hf_{n+2} \quad (10)$$

Derivation of Continuous Formulation of BSM for *K* = 3

Here, also the number of interpolation points, t = 1 and the number of collocation points, m = 4. Therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_n + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}\right] \quad (11)$$

Also the matrix D in (5) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_n + 2h & 3(x_n+h)^2 & 4(x_n+h)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 \\ 0 & 1 & 2x_n + 6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 \end{bmatrix} \quad (12)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the continuous scheme is also obtained using (2) and evaluating it at $x = x_{n+1}$, $x = x_{n+2}$ and $x = x_{n+3}$, the following discrete schemes are obtained:

$$y_{n+1} = y_n + \frac{3}{8}hf_n + \frac{19}{24}hf_{n+1} - \frac{5}{24}hf_{n+2} + \frac{1}{24}hf_{n+3}$$

$$y_{n+2} = y_n + \frac{1}{3}hf_n + \frac{4}{3}hf_{n+1} + \frac{1}{3}hf_{n+2}$$

$$y_{n+3} = y_n + \frac{3}{8}hf_n + \frac{9}{8}hf_{n+1} + \frac{9}{8}hf_{n+2} + \frac{3}{8}hf_{n+3} \quad (13)$$

Derivation of Continuous Formulation of BAMM for *K* = 4

For step number $k = 4$, number of interpolation points, $t = 1$ and the number of collocation points, m = 5. Therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_n + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\right] \quad (14)$$

Also the matrix D in (5) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 \end{bmatrix} \quad (15)$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the continuous scheme is also obtained using (2) and evaluating it at $x = x_{n+1}$, $x = x_{n+2}$, $x = x_{n+3}$ and $x = x_{n+4}$, the following discrete schemes are obtained:

$$\begin{cases} y_{n+1} = y_n + \frac{251}{720}hf_n + \frac{323}{360}hf_{n+1} - \frac{11}{30}hf_{n+2} + \frac{53}{360}hf_{n+3} - \frac{19}{720}hf_{n+4} \\ y_{n+2} = y_n + \frac{29}{90}hf_n + \frac{62}{45}hf_{n+1} + \frac{4}{15}hf_{n+2} + \frac{2}{45}hf_{n+3} - \frac{1}{90}hf_{n+4} \\ y_{n+3} = y_n + \frac{27}{80}hf_n + \frac{51}{40}hf_{n+1} + \frac{9}{10}hf_{n+2} + \frac{21}{40}hf_{n+3} - \frac{3}{80}hf_{n+4} \\ y_{n+4} = y_n + \frac{14}{45}hf_n + \frac{64}{45}hf_{n+1} + \frac{8}{15}hf_{n+2} + \frac{64}{45}hf_{n+3} + \frac{14}{45}hf_{n+4} \end{cases} \quad (16)$$

Convergence analysis

Here, the order, error constant, consistency and zero stability of the derived discrete schemes shall be investigated.

Order and Error Constant

The order and error constants of the discrete schemes in (10) are found in block form as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 - \beta_0 - \beta_1 - \beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 - \beta_1 - 2\beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 - \frac{1}{2}\beta_1 - 2\beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 = \begin{bmatrix} \frac{1}{24} \\ 0 \end{bmatrix}$$

Therefore, (10) has order, $p = 3$ and error constants $\frac{1}{24}, 0$

Similarly, the order and error constants of the discrete schemes in (13) are found in block form as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_0 - \beta_1 - \beta_2 - \beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 - \beta_1 - 2\beta_2 - 3\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{81}{40}\alpha_3 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 = \begin{bmatrix} -\frac{19}{720} \\ -\frac{1}{90} \\ -\frac{3}{80} \end{bmatrix}$$

Therefore, (13) has order, $p = 4$ and error constants $-\frac{19}{720}$, $-\frac{1}{90}$, $-\frac{3}{80}$

Also, the order and error constants of the discrete schemes in (16) are found in block form as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + 8\alpha_4 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 + \frac{32}{3}\alpha_4 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 - 8\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 + \frac{32}{3}\alpha_4 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 - \frac{32}{3}\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{81}{40}\alpha_3 + \frac{128}{15}\alpha_4 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 - \frac{32}{3}\beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 + \frac{81}{80}\alpha_3 + \frac{256}{45}\alpha_4 - \frac{1}{120}\beta_1 - \frac{4}{15}\beta_2 - \frac{81}{40}\beta_3 - \frac{128}{15}\beta_4 = \begin{bmatrix} \frac{3}{160} \\ \frac{1}{90} \\ \frac{3}{160} \\ 0 \end{bmatrix}$$

Therefore, (16) has order, $p = 5$ and error constants $\frac{3}{160}, \frac{1}{90}, \frac{3}{160}, 0$

Consistency

All the schemes in (10), (13) and (16) have their orders greater than one, hence the schemes are consistent, i.e. order $p \geq 1$.

Zero Stability

The zero stability of the discrete schemes in (10) is determined in a block form as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} \frac{2}{3} & -\frac{1}{12} \\ \frac{4}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & \frac{5}{12} \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

where $A_2^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B_2^{(1)} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{12} \\ \frac{4}{3} & \frac{1}{3} \end{bmatrix}$

$$\rho(\xi) = \det\left(\xi A_2^{(1)} - A_1^{(1)}\right) = 0$$

$$= \left|\xi A_2^{(1)} - A_1^{(1)}\right| = 0$$

Now we have

$$\rho(\xi) = \left|\xi\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right| = \begin{bmatrix} \xi & 0 \\ 0 & \xi \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \rho(\xi) = \begin{bmatrix} \xi & -1 \\ 0 & \xi - 1 \end{bmatrix}$$

Using Maple (17) software

$$\rho(\xi) = \xi(\xi - 1)$$

$$\Rightarrow \xi(\xi - 1) = 0$$

$\Rightarrow \xi_1 = 1, \xi_2 = 0$ .Since $|\xi_i| \leq 1, i = 1, 2$, then we observe that the discrete schemes in (10) satisfies the root condition and hence it is zero stable.

Similarly, the zero stability of the discrete schemes in (13) is determined in block form as follows:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{19}{24}&-\frac{5}{24}&\frac{1}{24}\\\frac{4}{3}&\frac{1}{3}&0\\\frac{9}{8}&\frac{9}{8}&\frac{3}{8}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}+h\begin{bmatrix}0&0&\frac{3}{8}\\0&0&\frac{1}{3}\\0&0&\frac{3}{8}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}$$

where $A_2^{(2)}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, A_1^{(2)}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}$ and

$$B_2^{(2)}=\begin{bmatrix}\frac{19}{24}&-\frac{5}{24}&\frac{1}{24}\\\frac{4}{3}&\frac{1}{3}&0\\\frac{9}{8}&\frac{9}{8}&\frac{3}{8}\end{bmatrix}$$

$$\rho(\xi)=\det\left(\xi A_2^{(2)}-A_1^{(2)}\right)=0$$

$$=\left|\xi A_2^{(2)}-A_1^{(2)}\right|=0$$

Now we have

$$\rho(\xi)=\left|\xi\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}\xi&0&0\\0&\xi&0\\0&0&\xi\end{bmatrix}-\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\right|$$

$$\Rightarrow\rho(\xi)=\begin{bmatrix}\xi&0&-1\\0&\xi&-1\\0&0&\xi-1\end{bmatrix}$$

Using Maple (17) software,

$$\rho(\xi)=\xi^2(\xi-1)$$

$$\Rightarrow\xi^2(\xi-1)=0$$

$\Rightarrow \xi_1=1,\xi_2=0,\xi_3=0$ .Since $|\xi_i|\le 1, i=1,2,3$, then we observe that the discrete schemes in (13) satisfies the root condition and hence it is zero stable.

Similarly, the zero stability of the discrete schemes in (13) is determined in block form as follows:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{323}{360}&-\frac{11}{30}&\frac{53}{360}&-\frac{19}{720}\\\frac{62}{45}&\frac{4}{15}&\frac{2}{45}&-\frac{1}{90}\\\frac{51}{40}&\frac{9}{10}&\frac{21}{40}&-\frac{3}{80}\\\frac{64}{45}&\frac{8}{15}&\frac{64}{45}&\frac{14}{45}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\end{bmatrix}+h\begin{bmatrix}0&0&0&\frac{251}{720}\\0&0&0&\frac{29}{90}\\0&0&0&\frac{27}{80}\\0&0&0&\frac{14}{45}\end{bmatrix}\begin{bmatrix}f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}$$

where $A_2^{(3)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, A_1^{(3)}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$ and

$$B_2^{(3)}=\begin{bmatrix}\frac{323}{360}&-\frac{11}{30}&\frac{53}{360}&-\frac{19}{720}\\\frac{62}{45}&\frac{4}{15}&\frac{2}{45}&-\frac{1}{90}\\\frac{51}{40}&\frac{9}{10}&\frac{21}{40}&-\frac{3}{80}\\\frac{64}{45}&\frac{8}{15}&\frac{64}{45}&\frac{14}{45}\end{bmatrix}$$

$$\rho(\xi)=\det\left(\xi A_2^{(3)}-A_1^{(3)}\right)=0$$

$$=\left|\xi A_2^{(3)}-A_1^{(3)}\right|=0$$

Now we have,

$$\rho(\xi)=\left|\xi\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}\xi&0&0&0\\0&\xi&0&0\\0&0&\xi&0\\0&0&0&\xi\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right|$$

$$\rho(\xi)=\begin{bmatrix}\xi&0&0&-1\\0&\xi&0&-1\\0&0&\xi&-1\\0&0&0&\xi-1\end{bmatrix}$$

Using Maple (17) software,

$$\Rightarrow \rho(\xi) = \xi^3(\xi - 1)$$

$$\Rightarrow \xi^3(\xi - 1) = 0$$

$\Rightarrow \xi_1 = 1, \xi_2 = 0, \xi_3 = 0, \xi_4 = 0$ .Since $|\xi_i| \leq 1, \mathrm{i} = 1, 2, 3, 4$, then we observe that the discrete schemes in (16) satisfies the root condition and hence it is zero stable.

Convergence

The block discrete schemes methods in (10), (13) and (16) are convergent since they are both consistent and zero-stable.

Stability analysis

The stability analyses of numerical methods for DDEs are considered. We considered on finding the P-stability and Q-stability of methods applied to the following DDEs of the form

$$\begin{aligned} y'(t) &= \lambda y(t) + \mu y(t-\tau), \quad t \geq t_0 \\ y(t) &= g(t), \quad t \leq t_0 \end{aligned} \tag{17}$$

where $g(t)$ is the initial function λ, μ are complex coefficients, $\tau = mh$, $m \in \mathbb{Z}^+$ and h is a step size or length. Let $H_1 = h\lambda$ and $H_2 = h\mu$, then from the discrete schemes in (10),

let

$$Y_{N+2} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, Y_N = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, F_{N+2} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}, F_N = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

and $F_N = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$

Since $A_2^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and

$$B_2^{(1)} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{12} \\ \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

we have, $$A_2^{(1)} Y_{N+2} = A_1^{(1)} Y_{N+1} + h \sum_{\mathrm{i}=1}^{2} B_i^{(1)} F_{N+i} \tag{18}$$

Also from the discrete schemes in (13),

let $$Y_{N+3} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}, Y_N = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, F_{N+3} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \text{ and } F_N = \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

Since

$$A_2^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B_2^{(2)} = \begin{bmatrix} \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{bmatrix}$$

we have, $$A_2^{(2)} Y_{N+2} = A_1^{(2)} Y_{N+1} + h \sum_{\mathrm{i}=1}^{2} B_i^{(2)} F_{N+i} \tag{19}$$

Also from the discrete schemes in (16),

let $$Y_{N+4} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix}, Y_{N\_} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, F_{N+4} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \text{ and } F_N = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

Since

$$A_2^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B_2^{(3)} = \begin{bmatrix} \frac{323}{360} & -\frac{11}{30} & \frac{53}{360} & -\frac{19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & -\frac{3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}$$

we have, $$A_2^{(3)} Y_{N+2} = A_1^{(3)} Y_{N+1} + h \sum_{\mathrm{i}=1}^{2} B_i^{(3)} F_{N+i} \tag{20}$$

According to Sirisena *et al.* (2019), the P- and Q-stability polynomials are obtained by applying (18) and (19) to (20). Thus the P-stability polynomials for the discrete schemes in (10), (13) and (16) are given respectively by

$$\psi^{(1)}(\xi) = \det\left[\left(A_2^{(1)} - H_1 B_2^{(1)}\right)\xi^{2+r} - \left(A_1^{(1)} - H_1 B_1^{(1)}\right)\xi^{1+r} - H_2 \sum_{i=1}^{2} B_i^{(1)} \xi^i\right]$$

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$$\psi^{(2)}(\xi) = \det\left[\left(A_2^{(2)} - H_1 B_2^{(2)}\right)\xi^{2+r} - \left(A_1^{(2)} - H_1 B_1^{(2)}\right)\xi^{1+r} - H_2\sum_{i=1}^{2} B_i^{(2)}\xi^i\right]$$

and

$$\psi^{(3)}(\xi) = \det\left[\left(A_2^{(3)} - H_1 B_2^{(3)}\right)\xi^{2+r} - \left(A_1^{(3)} - H_1 B_1^{(3)}\right)\xi^{1+r} - H_2\sum_{i=1}^{2} B_i^{(3)}\xi^i\right]$$

Also the Q-stability polynomials for the discrete schemes in (10), (13) and (16) are given respectively by

$$\pi^{(1)}(\xi) = \det\left[A_2^{(1)}\xi^{2+r} - A_1^{(1)}\xi^{1+r} - H_2\sum_{i=1}^{2} B_i^{(1)}\xi^i\right],$$

$$\pi^{(2)}(\xi) = \det\left[A_2^{(2)}\xi^{2+r} - A_1^{(2)}\xi^{1+r} - H_2\sum_{i=1}^{2} B_i^{(2)}\xi^i\right]$$

and

$$\pi^{(3)}(\xi) = \det\left[A_2^{(3)}\xi^{2+r} - A_1^{(3)}\xi^{1+r} - H_2\sum_{i=1}^{2} B_i^{(3)}\xi^i\right]$$

Using Maple 17 and MATLAB the P- and Q-stability region for the schemes in (10) and (13) and (16) are shown in Fig.1 to 6

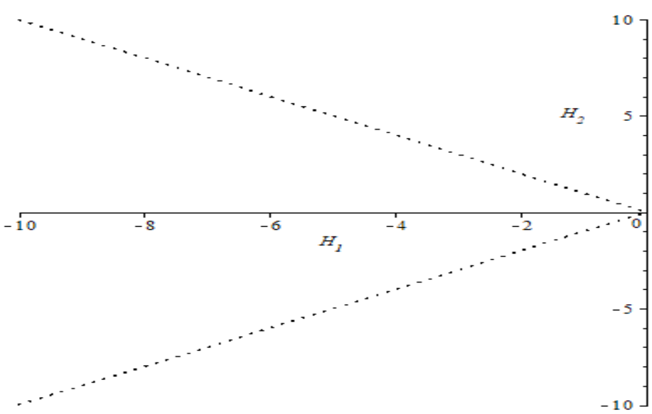

Fig.1. The P-stability region of the schemes in (10)

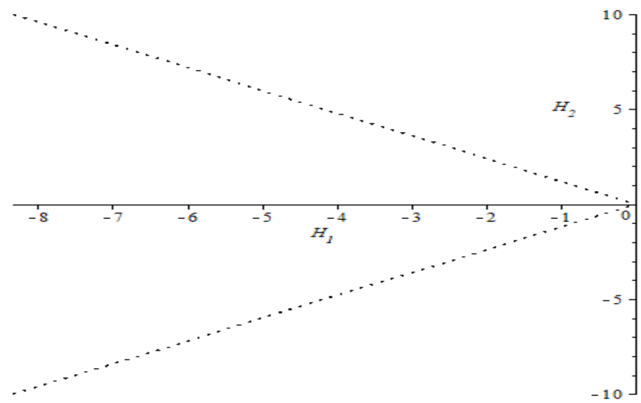

Fig.2. The P-stability region of the schemes in (13)

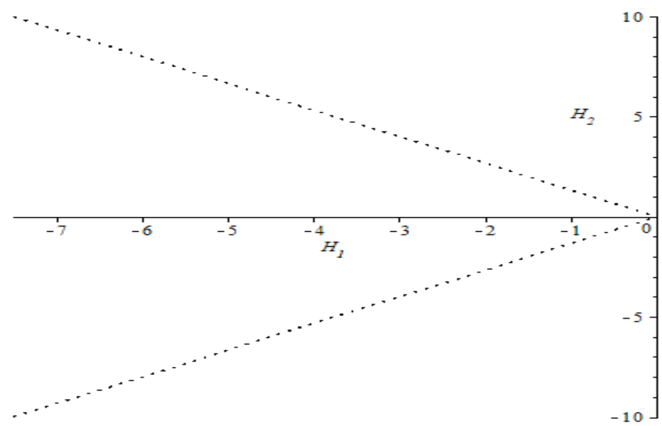

Fig.3.The P-stability region of the schemes in (16)

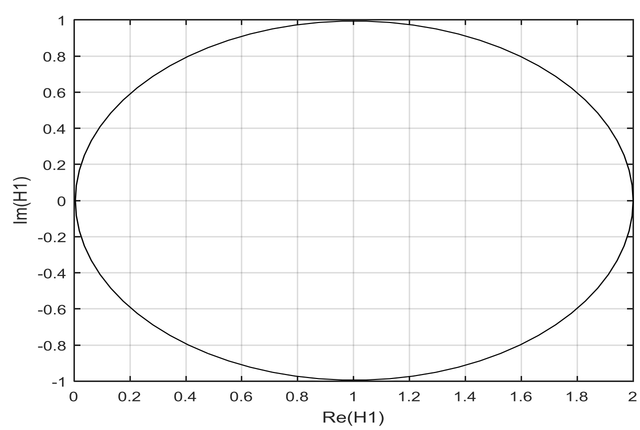

Fig.4.The Q-stability region of the schemes in (10)

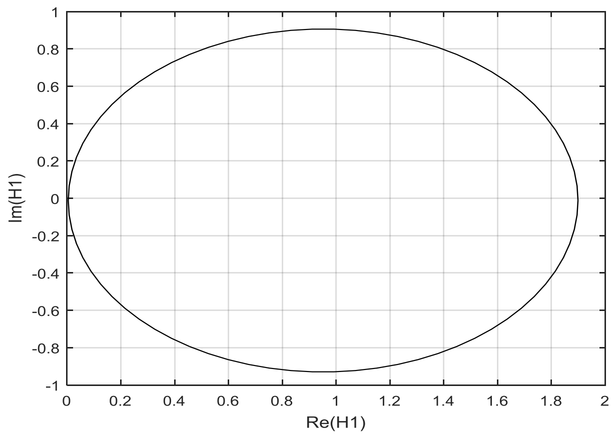

Fig.5.The Q-stability region of the schemes in (13)

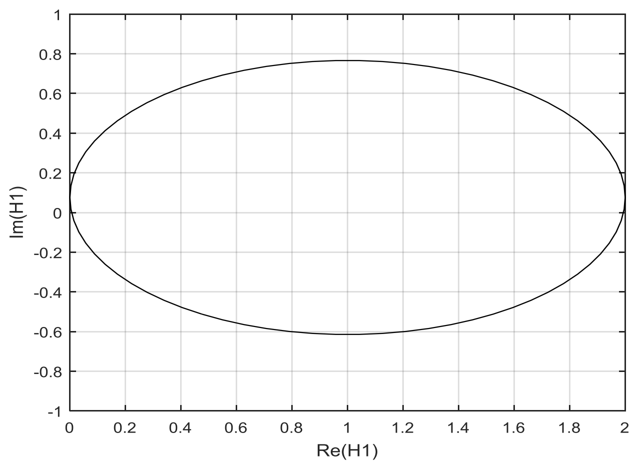

Fig.6.The Q-stability region of the schemes in (16)

From Figs.1 to 3, it is observed that the P-stability region of the schemes in (10) is about the same with that of the schemes in (13) and (16). Also from Figs 4 to 6, it is observed that the Q-stability region of the schemes in (10) is larger than that of the schemes in (13) and (16). The P-stability regions in Figs 1 to 3 lie inside the open ended region while in the Q-stability regions in Figs 4 to 6 lie inside the enclosed region.

Numerical Computations

In this section, some first order delay differential equations shall be solved using the block form of the discrete schemes been derived.

Numerical Examples

Example 1

$$y'(t) = -24y(t) - e^{(-25)}y(t-1),\ 0 \le t \le 3$$

$$y(t) = e^{(-25)t},t \le 0$$

Exact solution $y(t) = e^{(-25)t}$

Example 2

$$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}),$$
$$0 \le t \le 3$$

$$y(t) = 1 + e^{-3t},t \le 0$$

Exact solution $y(t) = 1 + e^{-3t}$

The above examples were solved using the schemes in (10), (13) and (16) as obtained by the Block Simpson's Methods and the results obtained were summarized in tables 1 to 2

Table 1: Solution of Example 1 using the BSM for Step Numbers k = 2, 3 & 4

t	Exact Solution	K=2 Numerical Solution	K=3 Numerical Solution	K=4 Numerical Solution
0.01	0.778800783	0.778687841	0.778785021	0.778798313
0.02	0.60653066	0.606556276	0.606527482	0.606529732
0.03	0.472366553	0.472318198	0.472347955	0.472364799
0.04	0.367879441	0.36791083	0.367857366	0.367880599
0.05	0.286504797	0.286487936	0.286491788	0.286504799
0.06	0.22313016	0.223159008	0.223112323	0.223130535
0.07	0.173773943	0.173771432	0.173756396	0.173773862
0.08	0.135335283	0.13535885	0.135323537	0.135336152
0.09	0.105399225	0.105402476	0.105386393	0.105399574
0.1	0.082084999	0.082103049	0.082073244	0.082085411
0.11	0.063927861	0.063932787	0.063919586	0.063928047
0.12	0.049787068	0.049800341	0.049778863	0.049787558
0.13	0.038774208	0.038779024	0.038766968	0.03877447
0.14	0.030197383	0.030206872	0.030192148	0.03019764
0.15	0.023517746	0.023521798	0.023512826	0.023517897
0.16	0.018315639	0.018322284	0.018311398	0.018315884
0.17	0.014264234	0.014267392	0.014261115	0.014264381
0.18	0.011108997	0.011113578	0.011106164	0.011109131
0.19	0.008651695	0.008654044	0.008649292	0.008651782

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0.2	0.006737947	0.006741066	0.006736159	0.006738062
0.21	0.005247518	0.005249211	0.005245933	0.005247592
0.22	0.004086771	0.004088874	0.004085442	0.004086836
0.23	0.003182781	0.003183973	0.003181783	0.003182825
0.24	0.002478752	0.002480158	0.002477883	0.002478804
0.25	0.001930454	0.00193128	0.001929731	0.001930489
0.26	0.001503439	0.001504373	0.001502894	0.001503469
0.27	0.00117088	0.001171444	0.00117041	0.001170901
0.28	0.000911882	0.000912498	0.000911494	0.000911905
0.29	0.000710174	0.000710556	0.00070988	0.00071019
0.3	0.000553084	0.000553489	0.000552834	0.000553098

Table 2: Solution of Example 2 using the BSM for Step Numbers k = 2, 3 & 4

t	Exact Solution	K=2 Num Soln	K=3 Num Soln	K=4 Num Soln
0.01	1.970445534	1.97044553	1.970445534	1.970445534
0.02	1.941764534	1.941764544	1.941764533	1.941764534
0.03	1.913931185	1.913931177	1.913931187	1.913931183
0.04	1.886920437	1.886920454	1.886920436	1.886920447
0.05	1.860707976	1.860707968	1.860707976	1.860707975
0.06	1.835270211	1.835270228	1.835270216	1.835270214
0.07	1.810584246	1.810584237	1.810584245	1.810584244
0.08	1.786627861	1.786627876	1.786627861	1.786627869
0.09	1.763379494	1.763379486	1.763379492	1.763379493
0.1	1.740818221	1.740818237	1.740818221	1.740818222
0.11	1.718923733	1.718923725	1.71892373	1.718923732
0.12	1.697676326	1.697676342	1.697676335	1.697676332
0.13	1.677056874	1.677056866	1.677056873	1.677056873
0.14	1.65704682	1.657046836	1.657046822	1.657046821
0.15	1.637628152	1.637628143	1.637628147	1.637628151
0.16	1.618783392	1.61878341	1.618783394	1.618783394
0.17	1.600495579	1.600495569	1.600495576	1.600495579
0.18	1.582748252	1.582748273	1.582748256	1.582748254
0.19	1.565525439	1.565525429	1.565525438	1.565525437
0.2	1.548811636	1.548811654	1.548811636	1.548811641
0.21	1.532591801	1.532591792	1.532591802	1.532591798
0.22	1.516851334	1.51685135	1.516851333	1.516851336
0.23	1.501576069	1.501576061	1.501576072	1.50157607
0.24	1.486752256	1.48675227	1.48675225	1.486752259
0.25	1.472366553	1.472366545	1.472366554	1.472366553
0.26	1.458406011	1.458406027	1.458406012	1.458406013
0.27	1.444858066	1.444858059	1.444858065	1.444858064
0.28	1.431710523	1.431710536	1.431710523	1.431710526
0.29	1.418951549	1.418951543	1.41895155	1.418951548
0.30	1.40656966	1.406569669	1.406569661	1.406569662

Results and Discussions

Here, the performances of the schemes derived in chapter two, shall be implemented in solving the examples in chapter four by computing their absolute errors.

Analysis of Results

The analysis of results is obtained by evaluating absolute difference of exact solutions and numerical solutions. The results are summarized in the tables 3 to 4;

Table 3: Absolute Errors of BSM for k = 2, 3 and 4 using Example 1

T	K=2 Error	k=3 Error	K=4 Error
0.01	0.000112942	1.57626E-05	2.47047E-06
0.02	2.56166E-05	3.17751E-06	9.27413E-07
0.03	4.83543E-05	1.85976E-05	1.75334E-06
0.04	3.13889E-05	2.20749E-05	1.15773E-06
0.05	1.68607E-05	1.30089E-05	1.83981E-09
0.06	2.88475E-05	1.7837E-05	3.75252E-07
0.07	2.51135E-06	1.7548E-05	8.14504E-08
0.08	2.35669E-05	1.17462E-05	8.69163E-07
0.09	3.25114E-06	1.28317E-05	3.49438E-07
0.1	1.80501E-05	1.17549E-05	4.12246E-07
0.11	4.92611E-06	8.2749E-06	1.85823E-07
0.12	1.32722E-05	8.20583E-06	4.89462E-07
0.13	4.81597E-06	7.2396E-06	2.62038E-07

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0.14	9.4883E-06	5.23581E-06	2.56708E-07
0.15	4.0523E-06	4.92E-06	1.50954E-07
0.16	6.64496E-06	4.24085E-06	2.45041E-07
0.17	3.15786E-06	3.11913E-06	1.47511E-07
0.18	4.58112E-06	2.83211E-06	1.34632E-07
0.19	2.34852E-06	2.40288E-06	8.71589E-08
0.2	3.1194E-06	1.78775E-06	1.1503E-07
0.21	1.69251E-06	1.5851E-06	7.38398E-08
0.22	2.1029E-06	1.32957E-06	6.49295E-08
0.23	1.19243E-06	9.97501E-07	4.41775E-08
0.24	1.40598E-06	8.69116E-07	5.18453E-08
0.25	8.25892E-07	7.2273E-07	3.46598E-08
0.26	9.33525E-07	5.45669E-07	2.9783E-08
0.27	5.64446E-07	4.69129E-07	2.08892E-08
0.28	6.16183E-07	3.8748E-07	2.27204E-08
0.29	3.81658E-07	2.94008E-07	1.56198E-08
0.3	4.04657E-07	2.50118E-07	1.32133E-08

Table 4: Absolute Errors of BSM for k = 2, 3 and 4 using Example 2

T	K=2 Error	K=3 Error	K=4 Error
0.01	3.54851E-09	4.51492E-10	4.51492E-10
0.02	1.04158E-08	5.84249E-10	4.15751E-10
0.03	8.27123E-09	1.72877E-09	2.27123E-09
0.04	1.72828E-08	7.17157E-10	1.02828E-08
0.05	8.42506E-09	4.25058E-10	1.42506E-09
0.06	1.65887E-08	4.58873E-09	2.58873E-09
0.07	8.97019E-09	9.70187E-10	1.97019E-09
0.08	1.49334E-08	6.65534E-11	7.93345E-09
0.09	8.33685E-09	2.33685E-09	1.33685E-09
0.1	1.63183E-08	3.18282E-10	1.31828E-09
0.11	8.43193E-09	3.43193E-09	1.43193E-09
0.12	1.5929E-08	8.92897E-09	5.92897E-09
0.13	8.49816E-09	1.49816E-09	1.49816E-09
0.14	1.61849E-08	2.18494E-09	1.18494E-09
0.15	8.62177E-09	4.62177E-09	6.21773E-10
0.16	1.81939E-08	2.19386E-09	2.19386E-09
0.17	9.81227E-09	2.81227E-09	1.87734E-10
0.18	2.0626E-08	3.62601E-09	1.62601E-09
0.19	9.69954E-09	6.99537E-10	1.69954E-09
0.2	1.7906E-08	9.40266E-11	4.90597E-09
0.21	9.0069E-09	9.93103E-10	3.0069E-09
0.22	1.55083E-08	1.4917E-09	1.5083E-09
0.23	8.06606E-09	2.93394E-09	9.33944E-10
0.24	1.404E-08	5.95997E-09	3.04003E-09
0.25	7.74101E-09	1.25899E-09	2.58985E-10
0.26	1.56948E-08	6.94776E-10	1.69478E-09
0.27	7.22294E-09	1.22294E-09	2.22294E-09
0.28	1.25709E-08	4.2908E-10	2.57092E-09
0.29	6.24764E-09	7.52361E-10	1.24764E-09
0.3	9.2594E-09	1.2594E-09	2.2594E-09

Conclusions

In conclusion, the discrete schemes of the BSM for step number k = 2, 3 and 4 respectively were deduced from their respective continuous formulations.

It was observed that, the block schemes are convergent, P-stable and Q-stable.

It was also observed in tables 3 to 4 that the BSM schemes for step number k = 4 performed better than the BSM schemes for step number k = 3 and BSM schemes for step number $k = 2$ respectively.

Recommendations
It is recommended that BSM schemes for k = 2, 3 and 4 are suitable for solving DDEs.
It is also recommended that the BSM schemes for a higher step number perform better than the BSM schemes of a lower step number.

Recommendations for Further Research

It is recommended that further research should be carried out on solving DDEs with BSM of higher step number because it gives more accurate results.

References

AI-mutib, A. N. (1977).Numerical methods for solving delay differential equations. Ph.D. Thesis. University of Manchester, United Kingdom.

Bocharov, G. A., Marchuk, G. I.and Romanyukha, A.A. (1996). Numerical solution by LMMs of stiff Delay Differential systems modeling an Immune Response. *Numer. Math.*, 73, 131-148.

Ishak, F., Majid, Z. A., & Suleiman, M. (2010). Two-point block method in variable step size technique for solving delay differential equations. *Journal of Materials Sci. and Eng.*, *4*(12), 86–90.

Ishak, F., Suleiman, M.B.,& Omar, Z. (2008). Two-point predictor-corrector block method for solving delay differential equations. *Matematika*,*24*(2), 131-140.

Majid, Z.A., Radzi, H.M.,& Ismail, F. (2013).Solving delay differential equations by the five-point one-step block method using Neville's interpolation. *International Journal of Computer Mathematics*.http://dx.doi.org/10.1080/00207160.2012. 754015.

Oberle, H.J.,&Pesh, H.J. (1981). Numerical treatment of delay differential equations by Hermite interpolation.*Numer. Math*,*37*, 235–255.

Sirisena, U. W. (1997). *A reformulation of the continuous general linear multistep method by matrix inversion for the first order initial value problems*. Ph.D. Thesis (Unpublished), University of Ilorin, Nigeria.

Sirisena, U. W., & Yakubu S. Y. (2019). Solving delay differential equation using reformulated backward differentiation methods. *Journal of Advances in Mathematics and Computer Science, 32*(2), 1 – 15.

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