Efficient Portfolio Management for a Commercial Bank under CEV Model

***[1]Akpanibah, E. E. and [2]Ini, U. O.**

[1]Department of Mathematics and Statistics, Federal University Otuoke, Nigeria
[2]Department of Mathematics and Computer Science, Niger Delta University, Nigeria

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Abstract

Considering the adverse effect of Corona virus on the financial market, there is need for banks to develop efficient portfolio strategy that is compact and takes into consideration the volatility of the stock market price. As a result of this, an efficient portfolio management for a commercial bank under constant elasticity of variance (CEV) model is studied using exponential utility function. A portfolio comprising of treasury security, marketable security and a loan is considered such that the last two assets are modelled by CEV model. Furthermore, the power transformation and change of variable technique are used to obtain explicit solutions of the optimal portfolio strategies, value function, bank's total assets, deposits and capital with numerical simulations. Finally, based on the investment strategies employ by the bank, the bank's asset is higher than that of its liability showing that the bank makes profit.

***Corresponding Author**: **Akpanibah, E. E.**; edikanakpanibah@gmail.com

Introduction

The stock market is recently in crisis due to the pandemic of the novel Corona virus (Covid-19). The commercial banks are among the financial institutions formed to accept money from their customers and give it out to others. They fulfil many functions which include receiving deposits from depositors, making payments upon the direction of its depositors, collecting funds from other banks payable to their customers, investing funds in securities for a return, safeguarding money, maintaining and servicing savings and checking accounts of their depositors, maintaining custodial accounts, i.e., accounts controlled by one person but for the benefit of another person and lending money (Grimsley, 2014). Following the rampaging effect of the corona virus on the world's economy, the commercial are the worst hit at this point due to the fact they depend heavily on the return on investments from the financial market, and are majorly out to make a profit. Nevertheless, for a commercial bank to make maximal profit, it has to give significant attention to how the bank's assets are managed. This include the amount of resources set out for investment (capital invested, retained earnings and deposits) and the bank's mind-set towards risk and return i.e how to distribute their resources among its assets for optimal returns. The theory of optimization is one of the instruments used in the banking sector and is a very important in finance to solve a wide range of stochastic optimization problems.

In banking, a number of authors have used the stochastic optimal control technique to solve optimization problems. Some of the authors who carried out research in this area include; Mukuddem-Petersen and Petersen (2006) who studied a problem related to the optimal risk management of banks in a stochastic dynamic setting. In particular, they studied the solution of an optimal stochastic control problem which minimizes bank market and capital adequacy risks by making choices about security allocation and capital requirements. Fouche *et al.* (2006) modelled non-risk-based and risk-based capital adequacy ratios. More specifically, they construct continuous-time stochastic models for the dynamics of the Leverage, Equity and Tier 1 ratios with the aim of deriving the CAR; also, they obtained an optimal asset allocation strategy for the Leverage Ratio on a given time interval. Mulaudzi *et al.* (2008) investigated the investment of bank funds in loans and treasuries with the aim of generating an optimal final fund level; In their work, they considered cases where the bank that takes behavioural aspects such as risk and regret into account. .In Witbooi *et al.* (2011), Cox-Huang technique was used to study commercial banking problem where the interest rate is of affine structure in a continuous-time; they solved the optimal capital allocation strategy that optimizes the banks total non-risk-weighted assets (TNRWAs) consisting of three assets namely a treasury, a marketable security and a loan. In Muller and Witbooi (2014), asset portfolio and capital adequacy management in banking was studied; they model a Basel III compliant commercial bank that operates in a financial market consisting of a treasury security, a marketable security, and a loan. Also, they considered the risk-free interest to be stochastic. Optimal portfolio strategy for banks whose shareholders have a power utility function and considers investment in a bank account, securities, and loans was investigated; they derived the solution to their problem by following the dynamic programming approach (Chakroun and

Abid, 2016). Muller (2018) studied optimal investment strategy and multiperiod deposit insurance pricing model for commercial banks; they considered investment in one risk free asset and two risky assets comprising of marketable security, and a loan such that the prices of the risky assets are modelled based on geometric Brownian motion.

The CEV model was first developed by (Cox and Ross, 1976) and is a natural extension geometric Brownian motion (GBM). According to Li *et al.* (2013), the model is capable of capturing the implied volatility skew. A lot of work has been studied on utility maximization under constant elasticity model in DC pension scheme (Xiao *et al.*, 2007; Gao, 2009). Gu (2010) studied optimal investment and reinsurance problem of utility maximization under CEV model. Li *et al.* (2013) investigated optimal investment problem with taxes, dividend and transaction cost using CEV model and logarithm utility function. Osu *et al.* studied optimal investment strategy with multiple contributors in a DC pension fund using Legendre transformation method. Li *et al.* (2017) studied the optimal investment problem for a DC pension plan with default risk and return of premiums clauses; they assumed they stock market price followed CEV model. Akpanibah and Ogheneoro (2018), studied the impact of additional voluntary contribution on the optimal investment strategy under CEV model; they used the power transformation method in solving their problem.

Our objective is to study the optimal portfolio strategies for a commercial bank under CEV model; here we consider a bank with investment portfolio comprising of treasury security, marketable security and a loan such that the risky assets follow the CEV model. Furthermore, we will solve for the bank assets, deposits and capital at time t.

Commercial Banking Model

Let consider a market with one risk free asset (treasury security) and two risky assets (marketable security and loan) such that is open continuously for a fixed time interval $[0,T]$, for $T > 0$ representing the expiring date of the investment. Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{W_0(t), W_1(t), W_2(t)\colon t \geq 0\}$ is a standard two dimensional Brownian motion and F is the filtration which represents the information generated by the two Brownian motions.

Let $\mathcal{A}(t)$ denote the price of the risk-free asset at time t and it is modelled as follows

$$\frac{d\mathcal{A}(t)}{\mathcal{A}(t)} = \mathcal{R}dt \quad \mathcal{A}(t) > 0 \tag{1}$$

Where $\mathcal{R} > 0$ and represent the risk-free interest rate.

Also, let $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ denote the prices of stock and loan respectively and their dynamics are given based on the stochastic differential equations as follows at $t \geq 0$

$$\frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} = (\mathcal{R} + c_1)dt + \delta_1 \mathcal{S}_1^{\beta}(t)dW_0(t) \tag{2}$$

$$\frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)} = \begin{pmatrix} (\mathcal{R} + c_2)dt + \delta_2 \mathcal{S}_2^{\beta}(t)dW_1(t) \\ +\delta_3 \mathcal{S}_2^{\beta}(t)dW_2(t) \end{pmatrix}, \tag{3}$$

where c_1, c_2, δ_1, δ_2 and δ_3 are positive and β represents the elasticity parameter. The loan is to be amortized over a period $[0,t]$. $W_0(t)$, $W_1(t)$ and $W_2(t)$ relate in such way $dW_0(t)dW_1(t) = dW_0(t)dW_2(t) = dW_1(t)dW_2(t) = 0$.
Let $\mathcal{X}(t)$ represent the bank's total asset at time $t \geq 0$, φ_0, φ_1 and φ_2 represent the proportion invested in treasury security, marketable security and a loan respectively. The dynamics of the bank's total asset at time $t \geq 0$ is given by the stochastic differential equation

$$d\mathcal{X}(t) = \left(\mathcal{X}(t) \begin{pmatrix} \varphi_0 \frac{d\mathcal{A}(t)}{\mathcal{A}(t)} + \\ \varphi_1 \frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} + \varphi_2 \frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)} \end{pmatrix} \\ +d\mathcal{C}(t) \right) \quad \mathcal{X}(0) = 1 \tag{4}$$

where $d\mathcal{C}(t)$ denote the rate of influx of the bank's capital whose dynamics is given as

$$d\mathcal{C}(t) = cdt, \ \mathcal{C}(0) > 0 \tag{5}$$

Let ρ_0, ρ_1 and ρ_2 represent the amount invested in each of the three assets such that

$$\rho_0 = \mathcal{X}(t)\varphi_0, \rho_1 = \mathcal{X}(t)\varphi_1 \text{ and } \rho_2 = \mathcal{X}(t)\varphi_2 \tag{6}$$

Substituting (1), (2), (3), (5) and (6), into (4), we have

$$d\mathcal{X}(t) = \begin{pmatrix} \begin{pmatrix} \mathcal{R}\rho_0 + \rho_1(\mathcal{R} + c_1) \\ +\rho_2(\mathcal{R} + c_2) + c \end{pmatrix} dt \\ +\rho_1\delta_1 \mathcal{S}_1^{\beta}(t)dW_0(t) + \rho_2\delta_2 \mathcal{S}_2^{\beta}(t)dW_1(t) \\ +\rho_2\delta_3 \mathcal{S}_2^{\beta}(t)dW_2(t) \end{pmatrix} \mathcal{X}(0) = 1 \tag{7}$$

Let φ be the optimal portfolio investment strategy and we define the utility attained by the investor from a given state x at time t as

$$\mathcal{G}_{\varphi}(t, s_1, s_2, x) = E_{\sup_{\varphi}}\left[\mathcal{V}(\mathcal{X}(T)) \mid \mathcal{S}_1(t) = s_1, \mathcal{S}_2(t) = s_2, \mathcal{X}(t) = x \right] \tag{8}$$

where$t$ is the timeand x is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as

$$\varphi^* \qquad \text{and } \mathcal{G}(t, s_1, s_2, x) = \sup_{\varphi} \mathcal{G}_{\varphi}(t, s_1, s_2, x) \tag{9}$$

Respectively such that

$$\mathcal{G}_{\varphi^*}(t, s_1, s_2, x) = \mathcal{G}(t, s_1, s_2, x).$$

The value function $\mathcal{G}_{\varphi}(t, s_1, s_2, x)$can be considered as a kind of utility function. The marginal utility of $\mathcal{G}_{\varphi}(t, s_1, s_2, x)$ is a constant, while the marginal utility of the original utility function $\mathcal{V}(\mathcal{X}(T))$ decreases to zero as $\mathcal{X}(T)\rightarrow\infty$ (Kramkov and Schachermayer, 1999). $\mathcal{G}_{\varphi}(t, s_1, s_2, x)$also inherits the convexity of $\mathcal{V}(\mathcal{X}(T))$ (Jonsson and Sircar, 2002). More precisely, it is strictly convex for $t < T$ even if $\mathcal{V}(\mathcal{X}(T))$ is not.

To understand the operation and management of a commercial bank, for a practical problem we study its stylized balance sheet, which records the assets (uses of funds) and liabilities (sources of funds) of the bank (Muller, 2018). The role of bank capital is to balance the assets and liabilities of the bank. A useful way, for our analysis, of representing the balance sheet of the bank is as follows:

$$R + S + L = D + B + C \qquad (10)$$

where R, S, L, D, B and C represent the values of reserves, securities, loans, deposits, borrowings and capital respectively. Each of the variables above is regarded as a stochastic process. The definitions of the terms above are can be found in (Muller and Witbooi, 2014; Muller, 2018). In order for a commercial bank to make a profit, it is important that the bank manages the asset side of its balance sheet properly.

The Optimization Program

Applying Ito's lemma and the maximum principle to (8), the Hamilton-Jacobi-Bellman (HJB) equation associated with (8) is given as

$$\left\{\begin{array}{c} \mathcal{G}_t + (\mathcal{R}+c_1)s_1\mathcal{G}_{s_1} + (\mathcal{R}+c_2)s_2\mathcal{G}_{s_2} + c\mathcal{G}_x \\ +\frac{1}{2}\delta_1^2 s_1^{2\beta+2}\mathcal{G}_{s_1s_1} + \frac{1}{2}(\delta_2^2+\delta_3^2)s_2^{2\beta+2}\mathcal{G}_{s_2s_2} \\ +\sup\left\{\begin{array}{c} \frac{1}{2}\left[\rho_1^2\delta_1^2 s_1^{2\beta} + \frac{1}{2}\rho_2^2(\delta_2^2+\delta_3^2)s_2^{2\beta}\right]\mathcal{G}_{xx} \\ +[r\rho_0+\rho_1(\mathcal{R}+c_1)+\rho_2(\mathcal{R}+c_2)]\,\mathcal{G}_x \\ +\rho_1\delta_1^2 s_1^{2\beta+1}\mathcal{G}_{s_1x} \\ +\rho_2(\delta_2^2+\delta_3^2)s_2^{2\beta+1}\mathcal{G}_{s_2x}\end{array}\right\}\end{array}\right\} = \qquad (11)$$

Differentiating (11) with respect to $\rho_1$and$\rho_2$and solving it we have

$$\rho_1{}^* = -\frac{\left[(\mathcal{R}+c_1)\,\mathcal{G}_x + \delta_1^2 s_1^{2\beta+1}\mathcal{G}_{s_1x}\right]}{\delta_1^2 s_1^{2\beta}\,\mathcal{G}_{xx}} \qquad (12)$$

$$\rho_2{}^* = -\frac{\left[(\mathcal{R}+c_2)\,\mathcal{G}_x + (\sigma_2^2+\sigma_3^2)s_2^{2\beta+1}\mathcal{G}_{s_2x}\right]}{(\sigma_2^2+\sigma_3^2)s_2^{2\beta}\,\mathcal{G}_{xx}} \qquad (13)$$

Substituting (12) and (13) into (11)

$$\left[\begin{array}{c} \mathcal{G}_t + (\mathcal{R}+c_1)s_1\mathcal{G}_{s_1} + (\mathcal{R}+c_2)s_2\mathcal{G}_{s_2} \\ +[\mathcal{R}\rho_0 + c]\mathcal{G}_x + \frac{1}{2}\delta_1^2 s_1^{2\beta+2}\left[\mathcal{G}_{s_1s_1} - \frac{\mathcal{G}_{s_1x}^2}{\mathcal{G}_{xx}}\right] \\ +\frac{1}{2}(\delta_2^2+\delta_3^2)s_2^{2\beta+2}\left[\mathcal{G}_{s_2s_2} - \frac{\mathcal{G}_{s_2x}^2}{\mathcal{G}_{xx}}\right] \\ -(\mathcal{R}+c_1)s_1\frac{\mathcal{G}_x\mathcal{G}_{s_1x}}{\mathcal{G}_{xx}} + (\mathcal{R}+c)s_2\frac{\mathcal{G}_x\mathcal{G}_{s_2x}}{\mathcal{G}_{xx}}\end{array}\right] \qquad (14)$$

Where, $\mathcal{G}(T, s_1, s_2, x) = \mathcal{V}(x)$ and $\mathcal{V}(x)$is the marginal utility of the bank. Next, we proceed to solve (14) for $\mathcal{G}$ considering a bank with exponential utility, then substitute the solution in (14) into (12) and (13)

Next, we consider a bank with exponential utility

$$\mathcal{V}(x) = -\frac{1}{n}e^{-nx} \qquad n > 0, \qquad (15)$$

The absolute risk aversion of a bank with the utility described in (15) is constant. We form a solution for (14) similar to the one in (Muller, 2018) with the form below:

$$\begin{cases}\mathcal{G}(t, s_1, s_2, x) = \frac{1}{n}e^{-nx+\mathcal{J}(t,s_1,s_2,)} \\ \qquad \mathcal{J}(T, s_1, s_2) = 0,\end{cases} \qquad (16)$$

Differentiating (16) w.r.t t, s_1, s_2, x, we have

$$\left.\begin{array}{c} \mathcal{G}_t = \mathcal{J}_t\mathcal{G},\ \mathcal{G}_{s_1} = \mathcal{J}_{s_1}\mathcal{G}\ ,\ \mathcal{G}_{s_2} = \mathcal{J}_{s_2}\mathcal{G}, \\ \mathcal{G}_{s_1s_1} = (\mathcal{J}_{s_1}^2 + \mathcal{J}_{s_1s_1})\mathcal{G}, \\ \mathcal{G}_{s_2s_2} = (\mathcal{J}_{s_2}^2 + \mathcal{J}_{s_2s_2}),\ \mathcal{G}_{xs_1} = -n\mathcal{J}_{s_1}\mathcal{G}, \mathcal{G}_{xs_2} = -n\mathcal{J}_{s_2}\mathcal{G} \\ \mathcal{G}_x = -n\mathcal{G}, \mathcal{G}_{xx} = n^2\mathcal{G}\end{array}\right\} \qquad (17)$$

Substituting (17) into (14), we have

$$\left[\begin{array}{c}\mathcal{J}_t - [\mathcal{R}\rho_0 + c]n + \frac{1}{2}\delta_1^2 s_1^{2\beta+2}\mathcal{J}_{s_1s_1} + \frac{(\mathcal{R}+c_1)^2}{2\sigma_1^2 s_1^{2\beta}} \\ +\frac{1}{2}(\delta_2^2+\delta_3^2)s_2^{2\beta+2}\mathcal{J}_{s_2s_2} + \frac{(\mathcal{R}+c_2)^2}{2(\delta_2^2+\delta_3^2)s_1^{2\beta}}\end{array}\right]\mathcal{G} = 0 \qquad (18)$$

We assume that since $\mathcal{G} \neq 0$, then

$$\left[\begin{array}{c}\mathcal{J}_t - [\mathcal{R}\rho_0 + c]n + \frac{1}{2}\sigma_1^2 s_1^{2\beta+2}\mathcal{J}_{s_1s_1} + \frac{(\mathcal{R}+c_1)^2}{2\delta_1^2 s_1^{2\beta}} \\ +\frac{1}{2}(\delta_2^2+\delta_3^2)s_2^{2\beta+2}\mathcal{J}_{s_2s_2} + \frac{(\mathcal{R}+c_2)^2}{2(\delta_2^2+\delta_3^2)s_1^{2\beta}}\end{array}\right] = 0 \qquad (19)$$

Taking the boundary condition $\mathcal{J}(T, s_1, s_2,) = 0$ into consideration, we find the solution to (19) as follows

Proposition 1: The solution of equation (19) is given as

$$\mathcal{J}(t, s_1, s_2,) = g_0 + g_1 s_1^{-2\beta} + g_2 s_2^{-2\beta}, \qquad (20)$$

where

$$\left[\begin{array}{c} g_0 = \left[\begin{array}{c}[\mathcal{R}\rho_0 + c]n(t-T) \\ -\frac{\beta(2\beta+1)}{2}\left[\begin{array}{c}(\mathcal{R}+c_1)^2 \\ +(\mathcal{R}+c_2)^2\end{array}\right]\left(t\left[T - \frac{t}{2}\right] - \frac{3T^2}{2}\right)\end{array}\right] \\ g_1 = \frac{(\mathcal{R}+c_1)^2}{2\delta_1^2}(T-t) \\ g_2 = \frac{(\mathcal{R}+c_2)^2}{2(\delta_2^2+\delta_3^2)}(T-t)\end{array}\right] \qquad (21)$$

Proof. Let

$$\begin{cases}\mathcal{J}(t, s_1, s_2,) = d_1(t,,p) + d_2(t,,q), p = s_1^{-2\beta}, q = s_2^{-2\beta} \\ \qquad d_1(T,p) = d_2(T,q) = 0\end{cases} \qquad (22)$$

Then

$$\left.\begin{array}{c} \mathcal{I}_t = d_{1t} + d_{2t},\ \mathcal{I}_{s_1} = -2\beta s_1^{-2\beta-1} d_{1p}, \\ \mathcal{I}_{s_1 s_1} = 2\beta(2\beta+1)s_1^{-2\beta-2} d_{1p} + 4\beta^2 s_1^{-4\beta-2} d_{1pp}, \\ \mathcal{I}_{s_2} = -2\beta s_2^{-2\beta-1} d_{2q}, \\ \mathcal{I}_{s_2 s_2} = 2\beta(2\beta+1)s_2^{-2\beta-2} d_{2q} + 4\beta^2 s_2^{-4\beta-2} d_{2qq} \end{array}\right\} \quad (23)$$

Substituting (23) into (19), we have

$$\begin{bmatrix} d_{1t} + d_{2t} - [\mathcal{R}\rho_0 + c]n + \frac{p(c_1+\mathcal{R})^2}{2\delta_1^2} + \frac{q(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)} \\ + \frac{1}{2}\delta_1^2[2\beta(2\beta+1)d_{1p} + 4\beta^2 p d_{1pp}] \\ + \frac{1}{2}(\delta_2^2+\delta_3^2)[2\beta(2\beta+1)d_{2q} + 4\beta^2 q d_{2qq}] \end{bmatrix} = 0 \quad (24)$$

Next, we assume a solution for equation (24) in the form

$$\begin{cases} d_1(t,,p) + d_2(t,q) = g_0 + p g_1 + q g_2 \\ g_0(T) = g_1(T) = g_2(T) = 0 \end{cases} \quad (25)$$

Differentiating (25) with respect to $t,\ p,,q$, we have

$$\begin{cases} d_{1t} + d_{2t} = g_{0t} + p g_{1t} + q g_{2t}, d_{1p} = g_1\ , \\ d_{1pp} = 0\ , d_{2q} = g_2\ , d_{2qq} = 0, \end{cases} \quad (26)$$

substituting equation (26) into (24), we have

$$\begin{bmatrix} g_{0t} - [\mathcal{R}\rho_0 + c]n \\ +\delta_1^2\beta(2\beta+1)g_1 + (\delta_2^2+\delta_3^2)\beta(2\beta+1)g_2 \\ +p\left(g_{1t} + \frac{(c_1+\mathcal{R})^2}{2\delta_1^2}\right) + q\left(g_{2t} + \frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)}\right) \end{bmatrix} = 0 \quad (27)$$

Splitting equation (27), we have

$$\begin{cases} [g_{0t} - [\mathcal{R}\rho_0 + c]k + \delta_1^2\beta(2\beta+1)g_1 \\ +(\delta_2^2+\delta_3^2)\beta(2\beta+1)g_2] = 0 \quad , \\ g_0(T) = 0 \end{cases} \quad (28)$$

$$\begin{cases} g_{1t} + \frac{(c_1+\mathcal{R})^2}{2\delta_1^2} = 0 \\ g_1(T) = 0 \end{cases}, \quad (29)$$

$$\begin{cases} g_{2t} + \frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)} = 0 \\ g_2(T) = 0 \end{cases}. \quad (30)$$

Solving the differential equations in (28), (29) and (30), we have

$$g_0 = \begin{bmatrix} [\mathcal{R}\rho_0 + c]n(t-T) \\ -\frac{\beta(2\beta+1)}{2}[(c_1+\mathcal{R})^2 + (c_2+\mathcal{R})^2]\left(t\left[T - \frac{t}{2}\right] - \frac{3T^2}{2}\right) \end{bmatrix}$$

$$g_1 = \frac{(c_1+\mathcal{R})^2}{2\delta_1^2}(T-t)$$

$$g_2 = \frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)}(T-t)$$

Hence completing the prove.

Proposition 2 The optimal value function is given as

$$\mathcal{G}(t, s_1, s_2, x) = \frac{1}{n}\exp\left(\begin{array}{c} -nx \\ + \begin{bmatrix} [v\rho_0 + c]n(t-T) \\ -\frac{\beta(2\beta+1)}{2}\begin{bmatrix} (c_1+\mathcal{R})^2 \\ +(c_2+\mathcal{R})^2 \end{bmatrix}\begin{pmatrix} t\left[T - \frac{t}{2}\right] \\ -\frac{3T^2}{2} \end{pmatrix} \\ + s_1^{-2\beta}\frac{(c_1+\mathcal{R})^2}{2\delta_1^2}(T-t) \\ + s_2^{-2\beta}\frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)}(T-t) \end{bmatrix} \end{array}\right) \quad (31)$$

Proof: From proposition 1,

$$\mathcal{I}(t, s_1, s_2,) = \begin{bmatrix} [\mathcal{R}\rho_0 + c]n(t-T) \\ -\frac{\beta(2\beta+1)}{2}[(c_1+\mathcal{R})^2 + (c_2+\mathcal{R})^2]\left(t\left[T - \frac{t}{2}\right] - \frac{3T^2}{2}\right) \\ + s_1^{-2\beta}\frac{(c_1+\mathcal{R})^2}{2\delta_1^2}(T-t) + s_2^{-2\beta}\frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)}(T-t) \end{bmatrix} \quad (32)$$

Also, from equation (16), we have

$$\mathcal{G}(t, s_1, s_2, x) = \frac{1}{n}e^{-nx + \mathcal{I}(t, s_1, s_2,)}$$

Substituting equation (32) into the above equation we have (31) which complete the proof.

Proposition 3 The optimal portfolio strategies for the three assets are given as

$$\varphi_0{}^* = 1 - \varphi_1{}^* - \varphi_2{}^* \quad (33)$$

$$\varphi_1{}^* = \frac{c_1+\mathcal{R}}{\chi n\delta_1^2 s_1^{2\beta}} \quad (34)$$

$$\varphi_2{}^* = \frac{c_2+\mathcal{R}}{\chi n(\delta_2^2+\delta_3^2)s_2^{2\beta}} \quad (35)$$

Proof: Recall from equation (12) and (13), the optimal amount invested in the two risky assets are given as

$$\rho_1{}^* = -\frac{\left[(c_1+\mathcal{R})\,\mathcal{G}_x + \delta_1^2 s_1^{2\beta+1}\mathcal{G}_{s_1 x}\right]}{\delta_1^2 s_1^{2\beta}\,\mathcal{G}_{xx}}$$

$$\rho_2{}^* = -\frac{\left[(c_2+\mathcal{R})\,\mathcal{G}_x + (\delta_2^2+\delta_3^2)s_2^{2\beta+1}\mathcal{G}_{s_2 x}\right]}{(\delta_2^2+\delta_3^2)s_2^{2\beta}\,\mathcal{G}_{xx}}$$

Also, from proposition 2, we have

$$\mathcal{G}_x = -n\mathcal{G}, \mathcal{G}_{xx} = n^2\mathcal{G},\ \mathcal{G}_{xs_1} = 0,\ \mathcal{G}_{xs_2} = 0 \quad (36)$$

Putting equation (36) into (12) and (13), we obtain

$$\rho_1{}^* = \frac{c_1+\mathcal{R}}{n\delta_1^2 s_1^{2\beta}} \quad (37)$$

$$\rho_2{}^* = \frac{c_2+\mathcal{R}}{n(\delta_2^2+\delta_3^2)s_2^{2\beta}} \quad (38)$$

from equation (6)

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$$\varphi_1{}^* = \frac{\rho_1{}^*}{\mathcal{X}}, \ \varphi_2{}^* = \frac{\rho_2{}^*}{\mathcal{X}} \tag{39}$$

substituting (39) into (37) and (38), we have

$$\varphi_1{}^* = \frac{c_1+\mathcal{R}}{\mathcal{X}n\delta_1^2 s_1^{2\beta}}, \ \varphi_2{}^* = \frac{c_2+\mathcal{R}}{\mathcal{X}n(\delta_2^2+\delta_3^2)s_2^{2\beta}}$$

which complete the proof.

Remark 1 If the elasticity parameter $\beta = 0$, equation (31), (33), (34) and (35) reduce to that of (Muller, 2018) as follows

$$\mathcal{G}(t, s_1, s_2, x) = \frac{1}{n} e^{-xn + \left[\substack{[\mathcal{R}\rho_0 + c]n(t-T) \\ +\frac{(c_1+\mathcal{R})^2}{2\delta_1^2}(T-t) + \frac{(c_2+\mathcal{R})^2}{2(\delta_2^2+\delta_3^2)}(T-t)}\right]}$$

$$\varphi_0 = 1 - \varphi_1 - \varphi_2, \quad \varphi_1 = \frac{c_1+\mathcal{R}}{\mathcal{A}n\delta_1^2} \text{ and } \varphi_2 = \frac{c_2+\mathcal{R}}{\mathcal{A}n(\delta_2^2+\delta_3^2)}$$

Let $\mathcal{D}(t)$ represent the bank's total liabilities which is given in (Muller, 2018) by the stochastic differential equation

$$\begin{cases} d\mathcal{D}(t) = c_3 dt + \delta_4 dW_3(t) \\ \mathcal{D}(0) = 0.8 \end{cases} \tag{40}$$

Assume $\mathcal{K}(t)$ is the bank's capital at time t, then $\mathcal{K}(t)$ is the difference between the bank's total assets and the bank's total liabilities at time t and is given as

$$\mathcal{K}(t) = \mathcal{X}(t) - \mathcal{D}(t) \tag{41}$$

From (41), we have

$$\begin{cases} d\mathcal{K}(t) = d\mathcal{X}(t) - d\mathcal{D}(t) \\ \mathcal{K}(0) = 0.2 \end{cases} \tag{42}$$

Solving equation (40), we have

$$\mathcal{D}(t) = 0.8 + \left(c_3 - \frac{1}{2}\delta_4^2\right)t + \delta_4 W_3(t) \tag{43}$$

Solving (7), we have

$$\mathcal{X}(t) = \begin{bmatrix} 1 + (\mathcal{R}+c)t + \frac{c_1(\mathcal{R}+c_1)}{n\delta_1^2}\int_0^t s_1^{-2\beta}(\tau)d\tau \\ + \frac{\mu_2(\mathcal{R}+\mu_2)}{n(\delta_2^2+\delta_3^2)}\int_0^t s_2^{-2\beta}(\tau)d\tau \end{bmatrix} \tag{44}$$

Solving equation (42), we have

$$\mathcal{K}(t) = \begin{pmatrix} 0.2 + (\mathcal{R}+c)t + \left(c_3 - \frac{1}{2}\delta_4^2\right)t \\ +\delta_4 W_3(t) + \frac{c_1(\mathcal{R}+c_1)}{n\sigma_1^2}\int_0^t s_1^{-2\beta}(\tau)d\tau \\ + \frac{\mu_2(r+c_2)}{n(\delta_2^2+\delta_3^2)}\int_0^t s_2^{-2\beta}(\tau)d\tau \end{pmatrix} \tag{45}$$

Numerical Simulations

In this section, we give numerical simulation of the optimal portfolio strategies when $\beta = 0$. To achieve this, the following data are used as in (Muller, 2018) unless otherwise stated. $\mathcal{R} = 0.065$, $c_1 = 0.035$, $c_2 = 0.045$, $c_3 = 0.12$, $\delta_1 = 0.08$, $\delta_2 = 0.095$, $\delta_3 = 0.065$, $\delta_4 = 0.15$, $n = 25$, $c = 0.0145$, $T = 12$

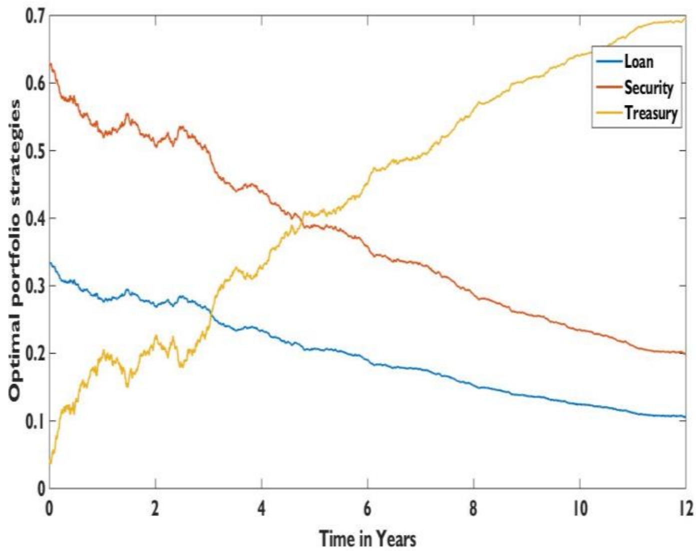

Fig.1: Time evolution of the optimal portfolio strategies

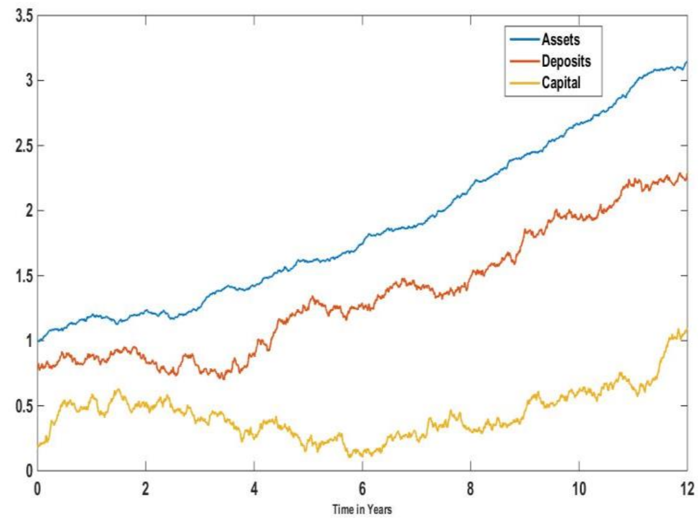

Fig. 2: Time evolution of Asset, Deposits and Capital

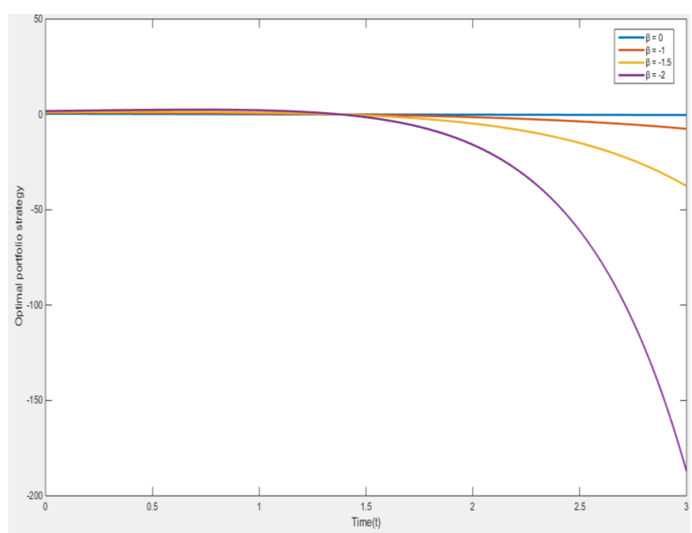

Figure 3: Time evolution of portfolio strategy with different β

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Discussion

The optimal portfolio strategies of the risky assets decrease continuously while that of the risk-free asset increase continuously. Similarly, the proportion of the bank's wealth invested in loan is smaller compared to that of loan; this could be due to the fact that loan is riskier than stock. This result is consistent with (Witbooi *et al*., 2011; Muller and Witbooi, 2014; Chakroun and Abid, 2016; Muller,2018). From figure 2, we observed that the based on the investment strategy implore by the bank, the bank's asset is higher than that of its liability showing that the bank makes profit other than loss. Figure 3, present a simulation of the optimal portfolio strategy against time with different values of the elasticity parameter β. The graph shows that as the elasticity parameter decreases, the bank is more scared to invest in risky asset as expiration date approaches. Furthermore, we observed a farther decrease when $\beta = -2$, showing how volatile the risky asset can be hence discouraging for investors with high risk aversion coefficient but when $\beta = 0$, the decline is almost unnoticeable, making the risky asset looks not volatile. This is shows that in developing an investment strategy, the geometric Brownian motion is not as good as the CEV model since it assume the volatility of the risky assets to be constant and may lead the bank astray while taking investment decisions especially in this critical times where the stock market is very unstable due to the crises rocking investment in almost every sectors of the economy as a result of the pandemic of the novel Corona virus (Covid-19). Furthermore, figure 3 shows that the CEV model is all encompassing as it has the ability to predict to a very high extent how volatile the risky assets are, thereby helping the bank to know how to appropriate their resources on the different assets available in the financial market.

Conclusion

In conclusion, the paper investigated optimal portfolio strategy for a commercial bank with exponential utility CEV model where a portfolio consisting of one risk free asset (treasury security) and two risky assets (marketable security and a loan) were considered such that the risky assets were modelled by CEV model. By using power transformation, change of variable approach, explicit solutions of the optimal portfolio strategies, value function, bank's total assets, deposits and capital were obtained.

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