International Journal of Basic Science and Technology, Vol. 4 No 1 (2018) p.01-11

The Exact Region for the Successful Application of Square Root Transformation in Time Series Decomposition using the Multiplicative Model

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Received:8th Jan.,2018
Accepted:3rd April, 2018
Available online
25th April, 2018

Keywords
Data transformation, Multiplicative time series model, Left-truncated - Normal distribution, Mean; Variance

Abstract: In time series decomposition using multiplicative model, the basic assumption required to proceed with a parametric time series analysis is that the error component (e_t) is normally distributed with unit mean and constant variance $(\sigma^2 < \infty)$. However, in practice there are cases where there are clear evidences of departure from normality and homogeneity of variance. As a remedial measure, such data sets require an appropriate transformation. Considering that successful transformation is one in which the original desired properties necessary for parametric data analysis are maintained after the transformation and considering that in this context, it is desired that the error component must be normally distributed with unit mean after a square root transformation, therefore the exact region in terms of the standard deviation, σ, for a successful square root transformation of a multiplicative time series model is investigated. From the study, it was established that normality and unit mean are actually achieved in the region $0 < \sigma \le 0.3$. Furthermore, the use of second order approximation for the infinite series obtained in the establishment of the functional expressions for the first and second moments of the square root transformed left truncated normal (1, σ^2) distribution was justified in this article. Finally, the established results were validated using a real-life data.

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Introduction

Transformations aim at improving the statistical analysis of time series data, by finding a suitable scale for which a model belonging to a simple and well known class, e.g. the normal regression model has the best option. An important class of transformations suitable for time series measured on a ratio scale with strictly positive support is the power transformation; originally proposed by Tukey (1957), as a device for achieving a model with simple structure, normal errors and constant error variance, It was subsequently modified by Box and Cox (1964).
Akpanta and Iwueze (2009) had also shown that Bartlett's transformation is also a power transformation and as a result of its recent popularity of application particularly in this area of study we shall also adopt it in this study. Data transformations are not only important in time series analysis but in all areas of statistical modeling where there are some basic assumptions required for the applications of the conventional methods of analysis. For instance, in the words of Ruppert(1999), "data transformations such as replacing a variable by its logarithm or by its square-root are used to simplify the structure of the data so that they follow a convenient statistical model" .Ruppert (1999) went further to say that "Transformations have one or more objectives including; (i) inducing a simple systematic relationship between a response and predictor variables in regression. (ii) stabilizing a variance, that is inducing a constant variance in a group of populations or in the residuals after a regression analysis and (iii) inducing a particular type of distribution, e.g. normal or symmetric distribution. The second and third goals as suggested by Ruppert (1999) are concerned with simplifying the "error structure" or random component of the data, however he also stated that transformations should not be used blindly to linearize a systematic relationship since if the errors of the data set before a transformation are homoscedastic then it is likely to be heteroscedastic after transformation. Also, the logarithm transformation can induce skweness and transform the data nearest to zero to outliers. Ruppert (1999) then concluded that any intelligent use of transformations requires that the effects of the transformations on the error structure be understood.
In situations where the assumptions required for parametric data analysis are violated, several options are

available (Sakia (1992): (i) Ignore the violation of the assumptions and proceed with the analysis as if all assumptions are satisfied. (ii) Decide what is the correct assumption in place of the one that is violated and use a valid procedure that takes into account the new assumption. (iii) Design a new model that has important aspects of the original model and satisfies all the assumptions, e.g. by applying a proper transformation to the data or filtering out some suspect data points which may be considered outlying. (iv) use a distribution-free procedure that is valid even if various assumptions are violated.

For more details on these listed options, see Graybill (1976, p. 213). Most researchers, however, have opted for (iii) which has attracted much attention as documented by Thoeni (1967) and Hoyle (1973) among others. In this study our interest would center on transformation as a remedy for situations where the assumptions for parametric data analysis are violated.

The Normal or Gaussian distribution is one of the most widely used of all random variables. Bell-shaped or approximately bell-shaped distributions are encountered in a large number of applications. Despite this utility, the fact that the values of a normally distributed random variable can in theory assume any value over the range $-\infty$ to ∞ may lead to significant computational errors in situations in which the distribution's outcomes are constrained. However, there are cases where this random variable is constrained to lie only on the negative or positive regions of the cartesian coordinates, which is often called the right or left truncation of the random variable. This restriction necessitated the derivation of distributions under the truncated environment.

Consider a normally distributed random variable X with a probability density function specified as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}, \; -\infty < x < \infty, \sigma^2 > 0 \qquad (1)$$

In many applications, the random variable X, which has a $N(1, \sigma^2)$ distribution do not admit values less than or equal to zero. We must therefore disregard or truncate all values of $X \leq 0$ to take care of the admissible region of $X > 0$ (Iwueze, 2007). Now if the values of X below or equal to zero cannot be observed due to censoring or truncation, then the resulting distribution is called the left truncated normal distribution with mean, 1 and constant variance, $\sigma^2 < \infty$ and whose Probability density function, $f^*(x)$ was obtained by Iwueze (2007) as

$$f^*(x) = \begin{cases} \dfrac{\exp\left\{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}, & 0 < x < \infty \\ 0, & -\infty < x \leq 0 \end{cases} \qquad (2)$$

with

$$E^*(X) = 1 + \frac{\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}\left[1 - \Phi\left(\frac{-1}{\sigma}\right)\right]} \qquad (3)$$

and

$$\mathbf{Var}^{*}\left(\mathbf{X}\right)=\frac{\boldsymbol{\sigma}^{2}}{2\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1+\mathbf{Pr}\left[\chi^{2}_{(1)}<\frac{1}{\sigma^{2}}\right]\right]-\frac{\sigma e^{-\frac{1}{2\sigma^{2}}}}{\sqrt{2\pi\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}}-\left[\frac{\sigma e^{-\frac{1}{2\sigma^{2}}}}{\sqrt{2\pi\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}}\right]^{2}. \qquad (4)$$

Conventionally a time series, X_t, $t \in Z$ is thought to comprise three components namely; the trend-cycle component (M_t), the seasonal component (S_t) and the irregular or error component (e_t). In the traditional time series decomposition, the multiplicative model is specified as

$$X_t = M_t * S_t * e_t,\ \ e_t \sim N(1, \sigma^2) \text{ and } \sum_{j=1}^{s} \mathbf{S}_{t+j} = \mathbf{s} \qquad (5)$$

where s is the seasonal periodic cycle of the series. After the estimation of the components, there is need to assess model adequacy by checking whether the model assumptions are satisfied. The component of the time series used for this assessment is the irregular component or the residual series, e_t. The basic assumption is that e_t is a Gaussian $N\left(1,\sigma_1^2\right)$ white noise. That is e_t's are uncorrelated random shocks with unit-mean and constant variance, $\sigma_1^2 < \infty$. For any fitted time series model, the residuals which are the error component are the estimates of this unobserved white noise. To check whether the error are normally distributed, one can construct a histogram of the standardized residuals and compare it with the standard normal distribution using the chi-square goodness of fit test or any other test of normality such as Tukey, Kolmogorov-Smirnov and Andersen Darling tests of normality and so on. To check whether the variance is constant, we examine the residual plot or use the appropriate test of homogeneity of variance such as Bartlett or Levene's test of homogeneity of variance. Finally, to check whether the residuals are white noise, we simply compute the sample autocorrelation and partial autocorrelation functions to check whether they do not form any pattern and are all statistically insignificant, within two standard deviations $\left(\pm\frac{2}{\sqrt{n}}\right)$ at 5% level of significance, where n is the number of observations in the data set.

Akpanta and Iwueze (2009) had shown how to apply Bartlett's transformation technique to time series data using the Buys-Ballot table without considering the time series model structure (Iwueze *et al*., 2011).

Akpanta and Iwueze (2009) showed that Bartlett's transformation for time series data is to regress the natural logarithms of the group or periodic standard deviations $\left(\hat{\sigma}_{i.}, i=1,2,\ldots,m\right)$ against the natural logarithms of the group or periodic means $\left(\bar{X}_{i.}, i=1,2,\ldots,m\right)$ and determine the slope, β of the relationship

$$\log_e \hat{\sigma}_{i.} = \alpha + \beta \log_e \bar{X}_{i.} + \text{error},\ i = 1, 2, \ldots m \qquad (6)$$

For non-seasonal data that require transformation, we split the observed time series X_t, $t=1,2,\ldots,n$ chronologically into m fairly equal different parts and compute $\left(\bar{X}_{i.}, i=1,2,\ldots,m\right)$ and $\left(\hat{\sigma}_{i.}, i=1,2,\ldots,m\right)$ for the parts. For seasonal data with the length of the periodic intervals, s, the Buys-Ballot table naturally partitions the observed data into m periods or rows for easy application. Akpanta and Iwueze (2009) also showed that Bartlett's transformation may also be regarded as the power transformation

$$Y_t = \begin{cases} \log_e X_t, \beta = 1 \\ X_t^{(1-\beta)}, \beta \neq 1 \end{cases} \qquad (7)$$

Summary of transformations for various values of β and data admissibility are given in Table 1.

There are various studies on the effects of transformation on the components of the multiplicative time series model in the statistical literature. The overall aim of such studies is to establish the conditions for successful transformations. A successful transformation except the logarithm transformation is achieved for the trend-cycle component when the original structure of the trend-cycle component remains unchanged after transformation. For example, in the same way that a linear trend-cycle component is expected to remain linear after transformation, a quadratic trend-cycle component should also remain quadratic after transformation. Furthermore, the seasonal indices of the transformed series are expected to be obtained directly by applying the chosen transformation on the original seasonal indices. Similarly the error component that is initially assumed to be $\mathbf{N}\left(1,\sigma_1^2\right)$ should also remain $\mathbf{N}\left(1,\sigma_2^2\right)$, even though σ_1^2 may or may not be equal to σ_2^2.

Iwueze *et al.* (2008) exhaustively studied the effect of the six popular transformations on the seasonal component of the multiplicative time series model and obtained the conditions for successful transformation. Iwueze *et al.* (2008). Iwu *et al*. (2009) had exhaustively studied the effect of the six popular transformations namely; logarithm, inverse, inverse-square, square-root, inverse-square-root and square on the trend-cycle components(for exponential, linear and quadratic cases) of the multiplicative time series model (5) and established the conditions for successful transformation .For more details see Iwu *et al*. (2009).

The effect of logarithm (Iwueze (2007)), square-root (Otuonye *et al*., (2011), inverse (Nwosu *et al*.,2013) and square (Ohakwe *et al.*,2013) transformations on the error component of the multiplicative time series model had been carried out. Here a successful transformation is achieved when the desirable properties/assumptions placed on the error component remains unchanged after transformation. The basic assumptions of interest for this study are; (i) Normality (ii) Unit mean and (iii) constant variance (which may or may not be equal to initial variance before transformation. Consequently Iwueze (2007) investigated the effect of logarithmic transformation on the error component (e_t) of a multiplicative time series model where ($e_t \sim N\left(1,\sigma^2\right)$) and discovered that the logarithm transform; Y = Log e_t can be assumed to be normally distributed with mean, zero and the same variance, σ^2 for $\sigma < 0.1$. Similarly Otuonye *et al*., (2011) and Nwosu *et al*., (2013) had studied the effects of square root and inverse transformations on the error component of the multiplicative time series model. Otuonye *et al*., (2011) discovered that the square root transform; $Y=\sqrt{e_t}$ can be assumed to be normally distributed with unit mean for $0<\sigma\leq 0.59$, where σ is the standard deviation of the original error component before transformation. Nwosu *et al.* (2013) discovered that the inverse transform $Y=\dfrac{1}{e_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma\leq 0.1$. Furthermore the condition for square transformation was obtained by Ohakwe *et al.* (2013) to be for $\sigma\leq 0.027$ .There is no question that Otuonye *et al*. (2011) had studied the effect of square root transformation on the error component of the multiplicative time series model and obtained the interval for successful transformation to be $\mathbf{0<\sigma\leq 0.59}$. However Otuonye *et al*. (2011) based their decision only on the computed values of $E^*(X)$ (mean of the left truncated N(1, σ^2) distribution) and E(Y) (mean of the square root transformed left truncated N(1, σ^2) distribution), where $E^*(X) = E(Y) = 1.0$. Secondly they encountered a sequence of infinite series in the derivations of the first and second moments and they simply used second order approximations of the series without any justification. Thirdly they failed to investigate the region or interval with regard to σ_1, where the bell-shaped characteristic (symmetrical curve about a unit mean) is satisfied for the square root transformed error component. It is on this note that we wish to appropriately reinvestigate the impact of square root transformation on the error component of the multiplicative time series model with the aim of finding the exact region of successful transformation.

The subsequent part of this paper is organized as follows; Section two contains the derivations of the functional expressions for the first and second moments of the square root transformed error component, though these had been done by Otuonye *et al*. (2011), there were significant gaps in their derivations needed to be filled up. In section three the exact region for the successful application of square root transformation in a multiplicative time series model would be established while a numerical demonstration of the results obtained using a real-life data would be contained in section four. The conclusion and references would be respectively contained in sections five and six

First and Second Moments of the Square Root Transformed Error Component

As was earlier stated in section one, Otuonye *et al.* (2011) had obtained the first and second moments by simply adopting a second order approximation of an infinite series encountered while deriving these moments without justifying its adoption. As a result of the above shortcoming we wish to appropriately re-establish these moments in this section.

Given that the pdf of the square root transformed error component had been obtained by Otuonye *et al.* (2011) which is specified as

$$f(y)=\frac{2y}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}e^{-\frac{1}{2}\left(\frac{y^2-1}{\sigma}\right)^2},y>0 \qquad (7)$$

we therefore obtain the mean denoted as E(Y), the second crude moment denoted as $E(Y^2)$ and the variance denoted as Var(Y) as follows;

we therefore obtain the mean denoted as E(Y), the second crude moment denoted as $E(Y^2)$ and the variance denoted as Var(Y) as follows;

$$\mathbf{E(Y) = \int_0^\infty y f(y) dy = \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_0^\infty y^2 e^{-\frac{1}{2}\left(\frac{y^2-1}{\sigma}\right)^2} dy} \tag{8}$$

Let

$$\mathbf{W = \frac{y^2-1}{\sigma}, -\frac{1}{\sigma} < u < \infty} \tag{9}$$

therefore,

$$y = (1+\sigma W)^{\frac{1}{2}} \text{ and } dy = \frac{\sigma dW}{2(1+\sigma W)^{\frac{1}{2}}} \tag{10}$$

Applying the substitution in (9) and its corresponding results given in (10) into (8), we obtain

$$\mathbf{E(Y) = \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^{\infty}(1+\sigma W)^{\frac{1}{2}} e^{-\frac{W^2}{2}} dW} \tag{11}$$

But from binomial theorem,

$$(\sigma W+1)^{\frac{1}{2}} = 1+\frac{\sigma W}{2}-\frac{\sigma^2 w^2}{8}+\frac{\sigma^3 w^3}{16}-\frac{5\sigma^4 w^4}{128}+\frac{7\sigma^5 w^5 e^{-\frac{w^2}{2}}}{256}\ \ldots \tag{12}$$

hence

$$E(Y) = \frac{1}{k}\int_{-\frac{1}{\sigma}}^{\infty}\left[1+\frac{\sigma w}{2}-\frac{\sigma^2 w^2}{8}+\frac{\sigma^3 w^3}{16}-\frac{5\sigma^4 w^4}{128}+\frac{7\sigma^5 w^5}{256}+\ldots\right]e^{-\frac{w^2}{2}} dw \tag{13}$$

where

$$k = \sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right] \tag{14}$$

In other to determine the appropriate interval for σ, we first obtain the interval $0<\sigma\leq b$ where the theoretical mean of the left-truncated $N(1,\sigma^2)$ distribution is approximately 1.0 to varying degrees of precision (that is the number of decimal places (dp)). To achieve this purpose we evaluate the expression for $E^*(X)$ as obtained by Iwueze (2007) for values of $\sigma=0.0001, 0.0002, 0.0003, \ldots, 0.5998, 0.5999, 0.6$. A series of 6000 values were obtained but for illustrative purposes, an abridged table of the results is given in Table 3(Appendix 1) while the summary of the results subject to the degree of precision is given in Table 2(Appendix 1). Also included in table 3 is the computations of variances of the left-truncated N(1, σ^2) and its square-root transform denoted as $Var^*(X)$ and Var(Y) respectively and the variance-ratio ($Var^*(X)$ / Var(Y). It is clear from table 3 that the variance-ratio is 4.0 as already obtained by Otuonye *et al.* (2011). This simply means that the variance of the left-truncated N(1, σ^2) is four times that of the square-root transformed left-truncated N(1, σ^2).

The points σ = 0.2561, 0.3019, 0.3822 and 0.5678 given in table 2(Appendix 1) are the critical values where the pdf curves of the transformed and the untransformed distributions would be investigated for bell-shaped characteristic of the normal curve. The outcome of this investigation would enable us to determine the degree of precision to adopt and thus the value of b. The curve shapes of the probability density functions of the left-truncated $N(1,\sigma^2)$ and that of the square root transformed left-truncated $N(1,\sigma^2)$ distributions for σ = 0.2561, 0.3019, 0.3822 and 0.5678 are given in figures 1 and 2. The purpose of this investigation is to determine the point where there is an indication of departure from normality (non-symmetrical curve about a unit mean). Based on the plots given in figures 1 and 2, there is no question that there is a clear evidence of departure from normality at $\sigma = 0.5678$,also there is a slight deviation from normality at $\sigma = 0.3822$ hence the interval of study in this paper where both distributions are almost perfectly normally distributed is $0 < \sigma \leq 0.3019$.

Furthermore, the integrals of the higher terms of the series starting from the fourth term are evaluated at σ = 0.3019. Based on the integral results given in Appendix 2, it is clear that these higher integrals contribute nothing to the value of the integral for the mean function and therefore the functional expression for the mean can be evaluated by ignoring them. Hence the mean function can be reasonably approximated using the second order, hence

$$E(Y) = \frac{1}{k}\int_{-\frac{1}{\sigma}}^{\infty}\left[1+\frac{\sigma w}{2}-\frac{\sigma^2 w^2}{8}\right]e^{-\frac{w^2}{2}}dw$$

$$= \frac{1}{k}\left[\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{w^2}{2}}dw + \frac{\sigma}{2}\int_{-\frac{1}{\sigma}}^{\infty} we^{-\frac{w^2}{2}}dw - \frac{\sigma^2}{8}\int_{-\frac{1}{\sigma}}^{\infty} w^2 e^{-\frac{w^2}{2}}dw\right] \quad (15)$$

For the first integral in (15), recall from Iwueze (2007) that

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{w^2}{2}}dw = 1-\Phi\left(-\frac{1}{\sigma}\right) \quad (16)$$

hence

$$\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{w^2}{2}}dw = \sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right] \quad (17)$$

For the second integral $\left(\frac{\sigma}{2}\int_{-\frac{1}{\sigma}}^{\infty} we^{-\frac{w^2}{2}}dw\right)$ in (15),

$$\text{let } w^2 = y, \frac{1}{\sigma^2} < y < \infty \quad (18)$$

then

$$w = y^{\frac{1}{2}} \quad (19)$$

and $dw = \frac{dy}{2w}$ (20)

thus

$$\frac{\sigma}{2}\int_{-\frac{1}{\sigma}}^{\infty} we^{-\frac{w^2}{2}}dw = \frac{\sigma}{4}\int_{\frac{1}{\sigma^2}}^{\infty} e^{-\frac{y}{2}}dy = \frac{\sigma}{4}\left[-2e^{-\frac{y}{2}}\right]_{\frac{1}{\sigma^2}}^{\infty} = \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{2} \quad (21)$$

Furthermore, using a similar result obtained by Iwueze (2007), it is easy to show that

$$\frac{\sigma^2}{8}\int_{-\frac{1}{\sigma}}^{\infty} w^2 e^{-\frac{w^2}{2}}dw = \frac{\sigma^2\sqrt{2\pi}}{16}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{8} \quad (22)$$

Combining the results in (17), (21) and (22), we obtain

$$E(Y) = 1 + \frac{\sigma}{8\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left\{\frac{5e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}} - \frac{\sigma}{2}\left(1+\Pr\left(\chi^2_{(1)} \leq \frac{1}{\sigma^2}\right)\right)\right\} \quad (23)$$

Derivation of the Variance of Y (Var (Y))

By definition

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

where

$$E(Y^2) = \int_0^{\infty} y^2 f(y)dy = \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\int_0^{\infty} y^3 e^{-\frac{1}{2}\left(\frac{y^2-1}{\sigma}\right)^2}dy$$

$$= \frac{2}{\sigma k}\int_0^{\infty} y^3 e^{-\frac{1}{2}\left(\frac{y^2-1}{\sigma}\right)^2}dy \quad (24)$$

The application of the substitution in (9) and its corresponding results in (10) into (24) yields

$$E(Y^2) = \frac{1}{k}\int_{-\frac{1}{\sigma}}^{\infty}(1+\sigma w)e^{-\frac{w^2}{2}}dw = \frac{1}{k}\left[\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{w^2}{2}}dw + \sigma\int_{-\frac{1}{\sigma}}^{\infty} we^{-\frac{w^2}{2}}dw\right]$$

….. (25)

$$\sigma\int_{-\frac{1}{\sigma}}^{\infty} we^{-\frac{w^2}{2}}dw = \sigma e^{-\frac{1}{2\sigma^2}} \quad (26)$$

Thus, substituting the results in (17) and (26) into (25) yields

$$E(Y^2) = \frac{1}{k}\left[\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right) + \sigma e^{-\frac{1}{2\sigma^2}}\right] = \left[1 + \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)}\right]$$

……… (27)

hence

$$\operatorname{Var}(Y) = 1 + \frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)} - \left\{1 + \frac{\sigma}{8\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left\{\frac{5e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}} - \frac{\sigma}{2}\left(1 + \Pr\left(\chi^2_{(1)} \le \frac{1}{\sigma^2}\right)\right)\right\}\right\}^2 \tag{28}$$

Considering that the left truncated N(1, σ^2) distribution and its square root counterpart are both normally distributed at $\sigma = 0.3019$ as evidenced in figures 1 and 2 and that the means of both distributions are approximately 1.0 as also evidenced by the results of the computations of E(X) and E(Y) (the abridged table given in table 3 of Appendix 1), we can confidently say that the interval for successful square root transformation when using a multiplicative time series model is $0 < \sigma \le 0.30$. To further confirm the point $\sigma = 0.30$, we tested for the normality of $\mathbf{Y} = \sqrt{\mathbf{X}^*}$ using the kolmogorov-smirnov (KS) test at 5% level of significance (α), where X^* are data generated from the N(1, σ^2) at σ = 0.28, 0.29, 0.3, 0.31, 0.32, 0.4. The results of the tests are presented in Table 4(Appendix 1). Based on the results, it is clear that normality of the generated data is rejected for $\sigma \ge 0.31$.

Application to Real-Life Data

In this section we would use data on the number of children immunized against BCG in Abia state from 2005-2009. The data is presented in Table 5:

For the analysis of the data in table 5, we did the following:

(i) justified the choice of the multiplicative model for data decomposition to obtain the residual series (error component (e_t)) of the original data (X_t))

(ii) calculated the mean and variance of (e_t)) and tested for its normality using Anderson-Darling test of normality.

(iii) justified the suitability of inverse-square-root transformation of the original data (X_t)

(iv) applied square-root transformation on X_t to obtain Y_t and decomposed Y_t to obtain the residual series (error component (e^*_t))

(v) calculated the mean and standard deviation of (e^*_t) and also tested itfor normality using Anderson-Darling test.

(vi) compared of (e_t) and (e^*_t)

The summary of the Descriptive Statistics of e_t and e^*_t resulting from the analysis is given in table 6. There is no question that the analysis clearly illustrates the effect of square root transformation on the error component of the multiplicative time series model in the sense that:

the square root transformation normalized the error component, both means are approximately 1.0 to the nearest whole number, the ratio of variance of e_t to that of e_t^* is approximately 4.0 as was established in the course of this study and the interval for successful application of the square root transformation for a multiplicative time series model is $0 < \sigma \le 0.30$

Conclusion

In this study, we established the interval $0 < \sigma \le 0.30$ as the exact region for the successful application of square root transformation when using a multiplicative time series model. We also justified the use of second order approximation of the infinite series encountered in deriving the functional expressions for the first and second moments of the square root transformed left truncated normal (1, σ^2) distribution. Finally, the established results were validated using a real life data.

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Appendix 1

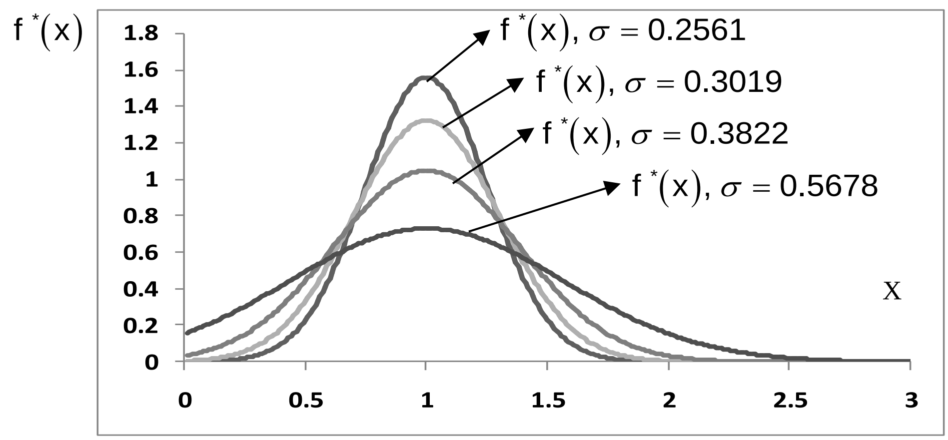

Figure 1: Curve Shapes of the Left-Truncated $N(1,\sigma^2)$

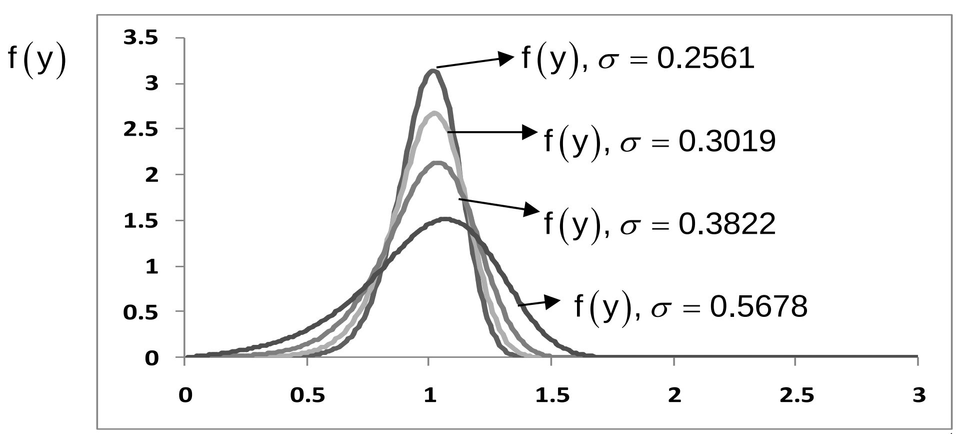

Figure 2: Curve Shapes of the Square root transformed Left-Truncated $N(1,\sigma^2)$

Table 1: Bartlett's Transformation for some values of β.

S/N	β	Required Transformation	Data admissibility
1	0	No transformation	$-\infty < X_t < \infty$
2	$\frac{1}{2}$	$\sqrt{X_t}$	$X_t > 0$
3	1	$\log_e X_t$	$X_t > 0$
4	$\frac{3}{2}$	$\frac{1}{\sqrt{X_t}}$	$X_t > 0$
5	2	$\frac{1}{X_t}$	$-\infty < X_t < \infty,\ X_t \neq 0$
6	3	$\frac{1}{X_t^2}$	$-\infty < X_t < \infty,\ X_t \neq 0$
7	-1	X_t^2	$-\infty < X_t < \infty$

Table 2: Summary of the results, where $\mathbf{E^*(X) = 1.0}$ subject to the required degree of

Sn	Degree of accuracy	Maximum value of σ
1	1 decimal place (dp)	0.5678
2	2 decimal place (dp)	0.3822
3	3 decimal place (dp)	0.3019
4	4 decimal place (dp)	0.2561

Table 3: An Abridged Table showing the Computations of $E^*(X)$, E(Y), $Var^*(X)$, Var(Y) and the Variance Ratio $\left(\frac{Var^*(X)}{Var(Y)}\right)$ for various values of σ

S/n	σ	$E^*(X)$	$E(Y)$	$Var^*(X)$	$Var(Y)$	$\frac{Var^*(X)}{Var(Y)}$
1	0.0001	1.0000	1.0000	0.0000000100	0.0000000025	4.0000
2	0.0002	1.0000	1.0000	0.0000000400	0.0000000100	4.0000
3	0.0003	1.0000	1.0000	0.0000000900	0.0000000225	4.0000
4	0.0004	1.0000	1.0000	0.0000001600	0.0000000400	4.0000
5	0.0005	1.0000	1.0000	0.0000002500	0.0000000625	4.0000
6	0.0006	1.0000	1.0000	0.0000003600	0.0000000900	4.0000
7	0.0007	1.0000	1.0000	0.0000004900	0.0000001225	4.0000
8	0.0008	1.0000	1.0000	0.0000006400	0.0000001600	4.0000
9	0.0009	1.0000	1.0000	0.0000008100	0.0000002025	4.0000
10	0.0010	1.0000	1.0000	0.0000010000	0.0000002500	4.0000
11	0.0011	1.0000	1.0000	0.0000012100	0.0000003025	4.0000
12	0.0012	1.0000	1.0000	0.0000014400	0.0000003600	4.0000
13	0.0013	1.0000	1.0000	0.0000016900	0.0000004225	4.0000
14	0.0014	1.0000	1.0000	0.0000019600	0.0000004900	4.0000
15	0.0015	1.0000	1.0000	0.0000022500	0.0000005625	4.0000
16	0.0016	1.0000	1.0000	0.0000025600	0.0000006400	4.0000
17	0.0017	1.0000	1.0000	0.0000028900	0.0000007225	4.0000

18	0.0018	1.0000	1.0000	0.0000032400	0.0000008100	4.0000
19	0.0019	1.0000	1.0000	0.0000036100	0.0000009025	4.0000
20	0.0020	1.0000	1.0000	0.0000040000	0.0000010000	4.0000
21	0.0021	1.0000	1.0000	0.0000044100	0.0000011025	4.0000
22	0.0022	1.0000	1.0000	0.0000048400	0.0000012100	4.0000
23	0.0023	1.0000	1.0000	0.0000052900	0.0000013225	4.0000
24	0.0024	1.0000	1.0000	0.0000057600	0.0000014400	4.0000
25	0.0025	1.0000	1.0000	0.0000062500	0.0000015625	4.0000
26	0.0026	1.0000	1.0000	0.0000067600	0.0000016900	4.0000
27	0.0027	1.0000	1.0000	0.0000072900	0.0000018225	4.0000
28	0.0028	1.0000	1.0000	0.0000078400	0.0000019600	4.0000
29	0.0029	1.0000	1.0000	0.0000084100	0.0000021025	4.0000
30	0.0030	1.0000	1.0000	0.0000090000	0.0000022500	4.0000
31	0.0031	1.0000	1.0000	0.0000096100	0.0000024025	4.0000
32	0.0032	1.0000	1.0000	0.0000102400	0.0000025600	4.0000
33	0.0033	1.0000	1.0000	0.0000108900	0.0000027225	4.0000
34	0.0034	1.0000	1.0000	0.0000115600	0.0000028900	4.0000
35	0.0035	1.0000	1.0000	0.0000122500	0.0000030625	4.0000
36	0.0036	1.0000	1.0000	0.0000129600	0.0000032400	4.0000

Table 4: Summary of Kolmogorov-Smirnov Test of Normality for the transformed series for the specified values of σ)

σ	KS value	Approx p-value	α	Decision
0.28	0.028	P > 0.15	0.05	Accept normality
0.29	0.042	P > 0.15	0.05	Accept normality
0.30	0.039	P > 0.15	0.05	Accept normality
0.31	0.089	0.01	0.05	Reject normality
0.32	0.055	0.036	0.05	Reject normality
0.40	0.067	0.01	0.05	Reject normality

Note: X^* is a data set of 300 values

Table 5: Number of children immunized against BCG in Abia state from 2005-2009.

Year	Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
2005	36	25	38	38	42	48	47	47	46	48	44	45
2006	57	61	47	35	28	40	42	39	42	40	42	46
2007	36	38	33	25	36	46	52	58	56	57	61	64
2008	80	84	103	109	110	111	109	117	116	118	117	120
2009	79	82	89	115	120	124	127	128	126	128	119	126

Table 6: Summary of the descriptive statistics of e_t and e_t^* for the data on the Number of children immunized against BCG in Abia state from 2005-2009

Error component	Mean	Variance (σ^2)	Standard Deviation (σ)	p-value of KS test	KS value	Level of significance	Decision	Variance ratio
e_t	0.9971	0.0599	0.2448	P < 0.01	0.135	0.05	Reject normality	3.84 ≈ 4.0
e_t^*	0.9992	0.0156	0.1249	0.063	0.112	0.05	Accept normality	

Appendix 2

Given the integral

$$\frac{\sigma^3}{16}\int_{-\frac{1}{\sigma}}^{\infty} w^3 e^{-\frac{w^2}{2}}\,dw \tag{A1}$$

let

$$w^2 = y,\ \frac{1}{\sigma^2} < y < \infty \tag{A2}$$

then

$$w = y^{\frac{1}{2}} \text{ and } dw = \frac{dy}{2w} \tag{A3}$$

therefore

$$\frac{\sigma^3}{16}\int_{-\frac{1}{\sigma}}^{\infty} w^3 e^{-\frac{w^2}{2}}\,dw = \frac{\sigma^3}{32}\int_{\frac{1}{\sigma^2}}^{\infty} y e^{-\frac{y}{2}}\,dy = \frac{\sigma^3}{32}\int_{\frac{1}{\sigma^2}}^{\infty} y^{\frac{4}{2}-1} e^{-\frac{y}{2}}\,dy$$

$$\frac{\sigma^3}{32}\left(4\Pr\left(\chi^2_{(4)} > \frac{1}{\sigma^2}\right)\right) = \frac{\sigma^3}{9}\left[1-\Pr\left(\chi^2_{(4)} \le \frac{1}{\sigma^2}\right)\right] = 0.0001\ \forall\ \sigma \le 0.3019 \tag{A4}$$

Similarly

$$\frac{5\sigma^4}{128}\int_{-\frac{1}{\sigma}}^{\infty} w^3 e^{-\frac{w^2}{2}}\,dw = \frac{5\sigma^4}{128}\sqrt{\frac{\pi}{2}}\left[1-\Pr\left(\chi^2_{(5)} \le \frac{1}{\sigma^2}\right)\right] = 0.0000\ \forall \sigma \le 0.3019 \tag{A5}$$

and

$$\frac{7\sigma^5}{256}\int_{-\frac{1}{\sigma}}^{\infty} w^5 e^{-\frac{w^2}{2}}\,dw = \frac{7\sigma^5}{32}\left[1-\Pr\left(\chi^2_{(6)} \le \frac{1}{\sigma^2}\right)\right] \le 0.0000\ \forall \sigma \le 0.3019 \tag{A6}$$